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Diagonal multi-matrix correlators and BPS operators in $\mathcal{N} = 4$ SYM

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Abstract: We present a complete basis of multi-trace multi-matrix operators that has a diagonal two point function for the free matrix field theory at finite N. This generalises to multiple matrices the single matrix diagonalisation by Schur polynomials. Crucially, it involves intertwining the gauge group $\mathrm{U}(N)$ and the global symmetry group $\mathrm{U}(M)$ with Clebsch-Gordan coefficients of symmetric groups S_n. When applied to $\mathcal{N} = 4$ super Yang-Mills we consider the $\mathrm{U}(3)$ subgroup of the full symmetry group. The diagonalisation allows the description of a dual basis to multi-traces, which permits the characterisation of the metric on operators transforming in short representations at weak coupling. This gives a framework for the comparison of quarter and eighth-BPS giant gravitons of $AdS_5 \times S^5$ spacetime to gauge invariant operators of the dual $\mathcal{N} = 4$ SYM.

Keywords: 1/N Expansion, AdS-CFT Correspondence, M(atrix) Theories, Gauge-gravity correspondence.

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Contents

1. Introduction

$\mathcal{N} = 4$ super-Yang Mills with $\mathrm{U}(N)$ gauge group has six $N \times N$ hermitian scalar matrices which can be organised into three complex scalar matrices. At zero Yang Mills coupling, the gauge invariant holomorphic functions of the matrices give rise to 1/8 BPS operators and by the operator-state correspondence, 1/8 BPS states. These states have a metric determined by the two-point function. The space of holomorphic operators can be organised into representations of a $\mathrm{U}(3)$ subgroup of the $\mathrm{SO}(6)$ R-symmetry. When we consider the analogous problem with a single complex matrix, we have $\mathrm{U}(1)$ symmetry, and the operators are half-BPS. The diagonalisation of the two-point function is accomplished by a basis of Schur polynomials [1]. Thanks to the AdS/CFT duality [2], this allows a map from gauge theory states to giant gravitons [3, 4] and LLM [5] geometries in AdS-spacetime. In this paper we will solve the analogous diagonalisation problem for the 1/4 and 1/8-BPS operators. The 1/4-BPS case involves a $\mathrm{U}(2)$ subgroup of the $\mathrm{U}(3)$.

We will consider, in general, a $\mathrm{U}(N)$ gauge theory with global $\mathrm{U}(M)$ symmetry. We have complex scalar fields which are valued in the adjoint representation of the gauge group $\mathrm{U}(N)$. The fields $(X_a)^i_j$ (where a is a fundamental $\mathrm{U}(M)$ index and i and j are $\mathrm{U}(N)$ indices) have a free field correlator with index structure

$$\left\langle (X_a)^i_j (X^\dagger_b)^k_l \right\rangle = \delta_{ab}\, \delta^i_l \delta^k_j \tag{1.1}$$

We have ignored the trivial spacetime dependence $|x_1 - x_2|^{-2}$ of the correlator, which is determined by conformal symmetry. While our primary interest is in the four-dimensional gauge theory, our main results have to do with the colour and flavour structure of the operators. Hence they are equally applicable to the reduced matrix models in one or zero dimension as long as the correlator (1.1) is valid.

We can build gauge-invariant polynomials in these fields by taking traces over the $\mathrm{U}(N)$ indices of polynomials in the X_a. For example if $M = 2$ we have fields X and Y and we can build operators such as $\mathrm{tr}(XXY)$ and $\mathrm{tr}(XX)\,\mathrm{tr}(Y)$. In this paper, we provide a complete linearly independent basis that spans these multi-trace multi-field operators. It has the correct counting of multi-trace multi-field operators given by Pólya theory. Furthermore it diagonalises the metric at zero coupling for finite N.

1.1 Summary of main results

We give here a summary of our key results. We find that a complete diagonal basis of multi-trace multi-matrix operators is given by operators of the form (3.28):

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{n!}\sum_{\alpha} B_{j\beta}\, S^{\tau,\Lambda}{}_{j}{}^{R}{}_{p}{}^{R}{}_{q}\;\; D^{R}_{pq}(\alpha)\,\mathrm{tr}(\alpha\, \mathbf{X}^{\mu})$$

The D^R_{pq} in this equation are matrix elements of the symmetric group on n letters, in the irreducible representation (irrep.) R. R also labels an irrep. of $\mathrm{U}(N)$; Λ labels an irrep. of $\mathrm{U}(M)$. $\mathbf{X}^{\mu}$ is an abbreviated notation for a tensor product of n matrix fields and the trace is being taken in $V^{\otimes n}$, where V is an N-dimensional vector space, the fundamental representation of $\mathrm{U}(N)$. The S factor is a Clebsch-Gordan coefficient coupling a tensor product of irreducible representations $R \otimes R$ of the symmetric group to the irrep. Λ. The B-factor is a branching coefficient for the restriction of the representation Λ of the symmetric group S_n to the trivial one-dimensional irrep. of its subgroup $S_{\mu_1} \times S_{\mu_2} \cdots \times S_{\mu_M}$. In the case at hand we are considering three complex matrices so $M = 3$. μ_1, μ_2, μ_3 give the numbers of the 3 complex matrices in the operator. The branching coefficients are explained in more detail in section 2.2. Useful formulae on relations between matrix elements and Clebsch-Gordan coefficients are collected in appendix A. The proof that the operators (3.28) provide a complete set of gauge invariant multi-matrix operators is given at the end of section 3.6. The diagonality of the two-point function

$$\left\langle \mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}\, \mathcal{O}^{\dagger\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2} \right\rangle = \delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta_{\beta_1\beta_2}\delta^{R_1R_2}\delta_{\tau_1\tau_2}\frac{|H_{\mu^{(1)}}|\mathrm{Dim}R_1}{d^2_{R_1}}$$

is derived as (3.25) in section 3.2.2. $|H_\mu|$ is the dimension $\mu_1!\mu_2!\cdots\mu_M!$ of the group $H_\mu \equiv S_{\mu_1} \times S_{\mu_2} \cdots \times S_{\mu_M}$.

As a prelude to the discussion of gauge invariant operators in section 3, we consider in section 2 the counting and two-point functions of untraced covariant operators. The mathematics of Schur-Weyl duality which relates representations of symmetric groups to those of unitary groups appears repeatedly in this paper. Section 2 uses the relation between irreps Λ of S_n, associated to a Young diagram Λ with n boxes, and the corresponding irrep of $\mathrm{U}(M)$. Section 3 uses the relation between irreps R of S_n and the corresponding irrep of $\mathrm{U}(N)$. Section 4 describes the extension to fermions, where the $\mathrm{U}(M)$ global symmetry is replaced by $\mathrm{U}(M_1|M_2)$. Section 5 describes the generalisation from two-point to multi-point correlators. We focus on extremal correlators which have non-renormalisation properties due to supersymmetry. Section 6 gives a general description of the finite N projector on multi-matrix multi-traces. While the diagonality (3.25) is derived at zero-coupling, thanks to supersymmetry, it has implications for weak and strong coupling physics. Section 7 describes the disentangling of operators which are in long representations at weak coupling from those that are genuine non-renormalised 1/8-BPS operators, and a characterisation of the metric on these operators. This allows a discussion of relations to the physics of giant gravitons at strong coupling and the related harmonic oscillator system. Since our key results depend only on the basic formula (1.1), they also apply to reduced matrix theories of multiple matrix fields in lower dimensions, such as 0 or 1. In the case of the reduced quantum mechanics considered in section 7.3, i.e 1-dimensional reduction, we discuss multiple commuting Hamiltonians related to the diagonal basis.

2. Gauge-covariant operators

Consider tensor products of operators of the form

$$\hat{\mathcal{O}}(\vec{a}) \equiv X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n} \tag{2.1}$$

forming words of length n. The a_i can take values from 1 to M. The operators correspond to states in $V_M^{\otimes n}$ where V_M is the fundamental representation of $\mathrm{U}(M)$; this dictates the action of $\mathrm{U}(M)$ on this operator. Their number is M^n.

There is also an action of the permutation group $\sigma \in S_n$ on these operators given by re-ordering

$$X_{a_{\sigma^{-1}(1)}} \otimes X_{a_{\sigma^{-1}(2)}} \otimes \cdots \otimes X_{a_{\sigma^{-1}(n)}} \tag{2.2}$$

If we recall that X_a is an $N \times N$ matrix, i.e a linear transformation of an N-dimensional space V, then the word (2.1) is a linear transformation of $V^{\otimes n}$. It follows that the permutation of operators can be achieved by a conjugation

$$X_{a_{\sigma^{-1}(1)}} \otimes X_{a_{\sigma^{-1}(2)}} \otimes \cdots \otimes X_{a_{\sigma^{-1}(n)}} = \sigma \cdot X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n} \cdot \sigma^{-1} \tag{2.3}$$

where the permutations are now acting on the n factors in $V^{\otimes n}$. In terms of indices, permuting the operators is equivalent to a simultaneous permutation of upper and lower indices. In the remainder of this section, we will be primarily interested in the $\mathrm{U}(M) \times S_n$ action on the operators.

Since the $\mathrm{U}(M)$ and S_n actions commute we have by Schur-Weyl duality (see for example [6, 7])

$$V_M^{\otimes n} = \oplus_\Lambda V_\Lambda^{\mathrm{U}(M)} \otimes V_\Lambda^{S_n} \tag{2.4}$$

The sum here is over Young diagrams Λ with n boxes and at most M rows that correspond to representations both of $\mathrm{U}(M)$ and S_n.

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The content of (2.4) is that there exists an invertible transformation $C^{\Lambda,m,i}_{a_1\cdots a_n}$ from the multi-index tensors to the irreducible representations of $\mathrm{U}(M)\times S_n$.

$$X_{a_1}\otimes X_{a_2}\otimes\cdots\otimes X_{a_n}=\sum_{\Lambda\vdash n}\sum_{i=1}^{d_\Lambda}\sum_{m=1}^{\mathrm{Dim}_M\Lambda} C^{\Lambda;\,i,m}_{a_1\cdots a_n}\,\hat{\mathcal{O}}^{\Lambda}_{i,m} \tag{2.5}$$

Here i labels the d_Λ states of the S_n representation Λ and m labels the $\mathrm{Dim}_M\Lambda$ states of the $\mathrm{U}(M)$ representation Λ.

The inverse is

$$\hat{\mathcal{O}}^{\Lambda}_{i,m}=\sum_{a_1\ldots a_n} C^{a_1,\ldots,a_n}_{\Lambda;\,i,m}\;X_{a_1}\otimes X_{a_2}\otimes\cdots\otimes X_{a_n} \tag{2.6}$$

2.1 Counting

Consider $\hat{\mathcal{O}}(\vec{a})$ as in (2.1), with $a_1,\cdots a_n$ chosen such that we have a fixed field content, i.e. fixed numbers of $X_1,X_2\ldots X_M$ given by $\mu_1,\mu_2\ldots\mu_M$. We will denote this set of integers as μ. Such operators can be written as

$$\hat{\mathcal{O}}^{\mu}(\sigma)\equiv\sigma\cdot X_1^{\otimes\mu_1}\otimes X_2^{\otimes\mu_2}\otimes\cdots\otimes X_M^{\otimes\mu_M}\cdot\sigma^{-1}=\sigma\,\mathbf{X}^{\mu}\,\sigma^{-1} \tag{2.7}$$

for some $\sigma\in S_n$. We have used the abbreviation $\mathbf{X}^{\mu}$ for $X_1^{\otimes\mu_1}\otimes X_2^{\otimes\mu_2}\otimes\cdots\otimes X_M^{\otimes\mu_M}$. Fixing $\vec{a}$ to take values from $1\cdots M$ determines μ and σ. The number of operators with fixed μ is

$$\frac{n!}{\mu_1!\mu_2!\cdots\mu_M!} \tag{2.8}$$

which is the number of ways of choosing n objects, with μ_1 of one kind, μ_2 of a second kind, and so on up to μ_M of the M'th kind. This is also the coefficient of $x_1^{\mu_1}\cdots x_M^{\mu_M}$ in the polynomial $(x_1+\cdots+x_M)^n$. From a group theory perspective we are counting operators (2.7) up to the symmetry

$$\sigma\,\mathbf{X}^{\mu}\,\sigma^{-1}\to\sigma h\,\mathbf{X}^{\mu}\,h^{-1}\sigma^{-1} \tag{2.9}$$

where the action of $h\in H_\mu=S_{\mu_1}\times S_{\mu_2}\cdots S_{\mu_M}$ leaves $\mathbf{X}^{\mu}$ unchanged. Thus we should quotient on the right by H_μ and count elements in the quotient group S_n/H_μ which has size

$$|S_n/H_\mu|=\frac{|S_n|}{|H_\mu|}=\frac{n!}{\mu_1!\mu_2!\cdots\mu_M!} \tag{2.10}$$

The words of fixed content μ form a vector space of dimension $\frac{|S_n|}{|H_\mu|}$. Now we want to understand the relation between the choice of μ and the decomposition (2.4) into Young diagrams. What is the dimension of the space of tensors of fixed Λ when we restrict the content to be specified by μ ? Equivalently this is the number of states in $V_\Lambda^{\mathrm{U}(M)}\otimes V_\Lambda^{S_n}$ which transform in the irrep. μ of the $\mathrm{U}(1)^M$ subgroup of $\mathrm{U}(M)$.

For a fixed irrep $V_\Lambda^{\mathrm{U}(M)}$ this number is

$$g([\mu_1],[\mu_2],\ldots[\mu_M];\Lambda)\equiv g(\mu;\Lambda) \tag{2.11}$$

This is the Littlewood-Richardson coefficient for the appearance of Λ in the tensor product of trivial single-row representations of $\mathrm{U}(M)$ $[\mu_1] \otimes \cdots \otimes [\mu_M]$. This is also known as the Kostka number and counts the number of $\mathrm{U}(M)$ states of Λ with field content μ. Since the irrep $V_\Lambda^{\mathrm{U}(M)}$ occurs in tensor space with multiplicity d_Λ, we have $\sum_\Lambda g(\mu;\Lambda) d_\Lambda$ states in $V_M^{\otimes n}$ with fixed content μ. A combinatoric implication is

$$\frac{n!}{\mu_1! \cdots \mu_M!} = \sum_\Lambda d_\Lambda g(\mu;\Lambda) = \frac{|S_n|}{|H_\mu|} \tag{2.12}$$

This is an identity which follows from properties of symmetric group representations.

2.2 Operators and correlators

Now we are going to implement the map (2.6) that takes us from $V_M^{\otimes n}$ to a state in $V_\Lambda^{\mathrm{U}(M)} \otimes V_\Lambda^{S_n}$ in order to write down a linearly independent basis of covariant operators. To understand where this basis comes from, we consider the two point function.

To work out the two point function we must reintroduce the $\mathrm{U}(N)$ indices. Each X_{a_i} is a complex matrix in $\mathrm{End}(V)$, where V is an N-dimensional vector space, the fundamental of $\mathrm{U}(N)$. Words of length n are elements of $\mathrm{End}(V^{\otimes n})$. We write the $\mathrm{U}(N)$ indices compactly as

$$(X_{a_1})^{i_1}_{j_1} \otimes \cdots \otimes (X_{a_n})^{i_n}_{j_n} = \mathbf{X}^I_J \tag{2.13}$$

Using the $\mathrm{U}(N)$ correlator (1.1) we get for the operator $\mathbf{X}^\mu = X_1^{\otimes \mu_1} \otimes \cdots \otimes X_M^{\otimes \mu_M}$

$$\langle (\mathbf{X}^\mu)^I_J (\mathbf{X}^{\dagger\mu})^K_L \rangle = \sum_{\gamma \in H_\mu} (\gamma)^I_L (\gamma^{-1})^K_J \tag{2.14}$$

The sum over $\gamma \in H_\mu = S_{\mu_1} \times \cdots \times S_{\mu_M}$ is over all the possible ways of contracting the X_1's, X_2's, etc. of the two operators.

Next consider the operator (2.7). Following the basic correlator (2.14) we find (see figure 1)

$$\begin{aligned} \left\langle (\hat{\mathcal{O}}^\mu(\sigma_1))^I_J \ \ (\hat{\mathcal{O}}^{\dagger\mu}(\sigma_2))^K_L \right\rangle &= \langle (\sigma_1 \mathbf{X}^\mu \sigma_1^{-1})^I_J (\sigma_2 \mathbf{X}^\mu \sigma_2^{-1})^K_L \rangle \\ &= \sum_{\gamma \in H_\mu} (\sigma_1 \gamma \sigma_2^{-1})^I_L (\sigma_2 \gamma^{-1} \sigma_1^{-1})^K_J \end{aligned} \tag{2.15}$$

It is useful to Fourier expand the group algebra in terms of matrix elements of irreducible representations, which are known to give a complete set of functions on the group by the Peter-Weyl theorem

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} = \frac{1}{n!} \sum_\sigma D^\Lambda_{ij}(\sigma) \hat{\mathcal{O}}^\mu(\sigma) \tag{2.16}$$

The sum σ is over all elements in S_n, and we are relating operators labelled by σ to operators labelled by (Λ, i, j). Here $D^\Lambda_{ij}(\sigma)$ is a representing matrix of the S_n representation Λ. i and j range over the states of this representation, going from 1 to the dimension d_Λ. In particular we choose orthogonal representing matrices obeying

$$D^\Lambda_{ij}(\sigma^{-1}) = D^\Lambda_{ji}(\sigma) \tag{2.17}$$

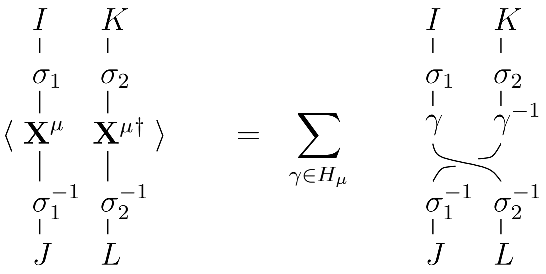

Figure 1: Two-point function for covariant operator.

such as the ones constructed in Hamermesh [8].

The two point function of the Fourier-transformed operators $\hat{\mathcal{O}}^{\Lambda\mu}_{ij}$ is

$$\begin{aligned}\left\langle (\hat{\mathcal{O}}^{\Lambda_1\mu^{(1)}}_{i_1j_1})^I_J(\hat{\mathcal{O}}^{\dagger\Lambda_2\mu^{(2)}}_{i_2j_2})^K_L\right\rangle &= \frac{\delta^{\mu^{(1)}\mu^{(2)}}}{(n!)^2}\sum_{\sigma_1,\sigma_2} D^{\Lambda_1}_{i_1j_1}(\sigma_1)\ D^{\Lambda_2}_{i_2j_2}(\sigma_2)\langle(\sigma_1\mathbf{X}^{\mu^{(1)}}\sigma_1^{-1})^I_J(\sigma_2\mathbf{X}^{\mu^{(2)}}\sigma_2^{-1})^K_L\rangle \\ &= \frac{\delta^{\mu^{(1)}\mu^{(2)}}}{(n!)^2}\sum_{\sigma_1,\sigma_2}\sum_{\gamma\in H_{\mu^{(1)}}} D^{\Lambda_1}_{i_1j_1}(\sigma_1)\ D^{\Lambda_2}_{i_2j_2}(\sigma_2)(\sigma_1\gamma\sigma_2^{-1})^I_L(\sigma_2\gamma^{-1}\sigma_1^{-1})^K_J \\ &= \frac{\delta^{\mu^{(1)}\mu^{(2)}}}{(n!)^2}\sum_{\sigma_1,\tau}\sum_{\gamma\in H_{\mu^{(1)}}} D^{\Lambda_1}_{i_1j_1}(\sigma_1)\ D^{\Lambda_2}_{i_2j_2}(\tau^{-1}\sigma_1\gamma)\ \ (\tau)^I_L(\tau^{-1})^K_J\end{aligned} \tag{2.18}$$

The diagonality in $\mu^{(1)},\mu^{(2)}$ follows easily from Wick's theorem and (1.1). In the last line we have replaced the σ_2 sum over S_n by a τ sum over S_n, where $\tau=\sigma_1\gamma\sigma_2^{-1}$. We can now expand out $D^{\Lambda_2}_{i_2j_2}(\tau^{-1}\sigma_1\gamma)$

$$\begin{aligned}\left\langle (\hat{\mathcal{O}}^{\Lambda_1\mu^{(1)}}_{i_1j_1})^I_J(\hat{\mathcal{O}}^{\dagger\Lambda_2\mu^{(2)}}_{i_2j_2})^K_L\right\rangle =& \frac{\delta^{\mu^{(1)}\mu^{(2)}}}{(n!)^2}\sum_{\sigma_1,\tau}\sum_{\gamma\in H_{\mu^{(1)}}} D^{\Lambda_1}_{i_1j_1}(\sigma_1)\ D^{\Lambda_2}_{i_2a}(\tau^{-1})\times \\ &\times D^{\Lambda_2}_{ab}(\sigma_1)D^{\Lambda_2}_{bj_2}(\gamma)\ (\tau)^I_L(\tau^{-1})^K_J \\ =&\delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\frac{|H_{\mu^{(1)}}|}{n!d_{\Lambda_1}}\ D^{\Lambda_1}_{j_1j_2}(\Gamma)\sum_{\tau} D^{\Lambda_1}_{i_1i_2}(\tau)\ (\tau)^K_J(\tau^{-1})^I_L\end{aligned} \tag{2.19}$$

We have defined $\Gamma=\frac{1}{|H_{\mu^{(1)}}|}\sum_{\gamma\in H_{\mu^{(1)}}}\gamma$, where $|H_{\mu^{(1)}}|=\mu_1^{(1)}!\cdots\mu_M^{(1)}!$. In the second line we performed the sum over σ_1 using identity (A.3) from appendix section A on formulae.

$D^{\Lambda}_{jk}(\Gamma)$ is a projector from the representation space of Λ onto the subspace which is invariant under $H_\mu\equiv H_{\mu^{(1)}}=H_{\mu^{(2)}}$. The irreducible representation Λ of S_n gives, by restriction, a representation of H_μ, generally reducible. One can decompose it in terms of irreps of H_μ. The projector Γ picks out the trivial irrep $\mathbf{1}(H_\mu)$ in this. This occurs with a multiplicity $g(\mu;\Lambda)$. We can write $D^{\Lambda}_{j_1j_2}(\Gamma)=\langle\Lambda,j_1|\Gamma|\Lambda,j_2\rangle$ as

$$\langle\Lambda,j_1|\Gamma|\Lambda,j_2\rangle=\sum_{\beta}\langle\Lambda,j_1|\Lambda(S_n)\to\mathbf{1}(H_\mu);\beta\rangle\ \langle\Lambda(S_n)\to\mathbf{1}(H_\mu),\beta|\Lambda,j_2\rangle \tag{2.20}$$

The index β runs over an orthonormal basis for the multiplicity of the trivial irrep. $\mathbf{1}(H_\mu)$. Using the orthonormality of β and inserting a complete set of states

$$\begin{aligned}\delta_{\beta_1\beta_2} &= \langle \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta_1 | \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta_2 \rangle \\ &= \sum_{j=1}^{d_\Lambda} \langle \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta_1 | \Lambda, j \rangle \, \langle \Lambda, j | \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta_2 \rangle \end{aligned} \tag{2.21}$$

This gives an orthogonality relation for the *branching coefficients* $\langle \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta | \Lambda, j \rangle$. From the reality of the symmetric group irreps.

$$\langle \Lambda, j | \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta \rangle = \langle \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta | \Lambda, j \rangle \tag{2.22}$$

See [9] and appendix D for calculations of these branching coefficients. To save space we shall define

$$B_{j\beta} \equiv \langle \Lambda, j | \Lambda(S_n) \to \mathbf{1}(H_\mu), \beta \rangle \tag{2.23}$$

It should be clear from the context which Λ and μ are being used.

Now we use these orthogonality properties to define

$$\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta} = \sum_j B_{j\beta} \, \hat{\mathcal{O}}^{\Lambda\mu}_{ij} \tag{2.24}$$

Note that the number of operators $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ is $d_\Lambda g(\mu;\Lambda)$ since i runs over d_Λ values and β runs over $g(\mu;\Lambda)$ values. This therefore correctly counts the covariant operators and gives the explicit map (2.6) from $V_M^{\otimes n}$ to a state in $V_\Lambda^{\mathrm{U}(M)} \otimes V_\Lambda^{S_n}$ for which we have been looking,

$$\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta} = \sum_j \; B_{j\beta} \, \frac{1}{n!} \sum_\sigma D^\Lambda_{ij}(\sigma) \, \sigma \, \mathbf{X}^\mu \, \sigma^{-1} \tag{2.25}$$

Note that together μ and β give us the $\mathrm{U}(M)$ state m. Furthermore, using equations (2.19) (2.21) and (2.23) we see that these operators $\mathcal{O}^{\Lambda\mu}_{i\beta}$ have a simple 2-point function

$$\left\langle (\hat{\mathcal{O}}^{\Lambda_1\mu^{(1)}}_{i_1\beta_1})^I_J (\hat{\mathcal{O}}^{\dagger\Lambda_2\mu^{(2)}}_{i_2\beta_2})^K_L \right\rangle = \delta^{\mu^{(1)}\mu^{(2)}} \delta^{\Lambda_1\Lambda_2} \delta_{\beta_1\beta_2} \frac{|H_{\mu^{(1)}}|}{n! d_{\Lambda_1}} \sum_\tau \; D^{\Lambda_1}_{i_1i_2}(\tau)(\tau)^K_J (\tau^{-1})^I_L \tag{2.26}$$

2.3 Basis from projection

We can also understand our basis as a projection onto a linearly-independent subspace of the $\hat{\mathcal{O}}^{\Lambda\mu}_{ij}$. The general theory is outlined in appendix B.

The set of operators $\hat{\mathcal{O}}^{\Lambda\mu}_{ij}$ appear to be giving d_Λ^2. We know that the dimension of the representation Λ with field content μ is instead $d_\Lambda g(\mu;\Lambda)$. The operators $\hat{\mathcal{O}}^{\Lambda\mu}_{ij}$ are not in fact linearly independent. This is because permutations in H_μ leave the $\mathbf{X}^\mu$ fixed. For any permutation $\gamma \in H_\mu$, we have $\mathcal{O}^\mu(\sigma) = \mathcal{O}^\mu(\sigma\gamma)$. For the Fourier transform, we have by relabelling $\sigma \to \sigma\gamma$ and using $D^\Lambda_{ij}(\sigma\gamma^{-1}) = \sum_k D^\Lambda_{ik}(\sigma) D^\Lambda_{jk}(\gamma)$, that

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} = D^\Lambda_{jk}(\gamma) \hat{\mathcal{O}}^{\Lambda\mu}_{ik} \qquad \forall \gamma \in H_\mu \tag{2.27}$$

Here the j index belongs to the representation Λ and this equation says that we seek the subspace which is invariant under the subgroup H_μ.

So we seek operators, $\mathcal{O}^{\Lambda\mu}_{ij}$ such that

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} = \sum_k \hat{\mathcal{O}}^{\Lambda\mu}_{ik} D^{\Lambda}_{jk}(\Gamma) \tag{2.28}$$

which, using equations (2.20) (2.22) (2.23), becomes

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} = \sum_{k,\beta} \hat{\mathcal{O}}^{\Lambda\mu}_{ik} B_{j\beta} B_{k\beta} \tag{2.29}$$

This is precisely the situation outlined in B. According to the analysis there, a basis for such operators is therefore given by

$$\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta} = \sum_j B_{j\beta}\, \mathcal{O}^{\Lambda\mu}_{ij} \tag{2.30}$$

which is exactly the operator we found to have nice diagonality properties for the two point function.

2.4 Recovering the permutation basis

Given the map (2.25) from operators $\hat{\mathcal{O}}^{\mu}(\sigma)$ of (2.7) labelled by permutations, to operators $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ labelled by irreps Λ and states i, j, we would also like to invert it, i.e. to provide the map (2.5) from $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ to $\hat{\mathcal{O}}^{\mu}(\sigma)$. From (2.29) we have

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} = \sum_\beta B_{j\beta} \hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}\,, \tag{2.31}$$

so we simply have to find the inverse of the Fourier transform (2.16) to give $\hat{\mathcal{O}}^{\mu}(\sigma)$ in terms of $\hat{\mathcal{O}}^{\Lambda\mu}_{ij}$

$$\hat{\mathcal{O}}^{\mu}(\sigma) = \sum_\Lambda \sum_{i,j} d_\Lambda D^{\Lambda}_{ij}(\sigma) \hat{\mathcal{O}}^{\Lambda\mu}_{ij} \tag{2.32}$$

This can be shown by using $\sum_{i,j} D^{\Lambda}_{ij}(\sigma_1) D^{\Lambda}_{ij}(\sigma_2) = \chi_\Lambda(\sigma_1\sigma_2^{-1})$ and that the expansion of the delta-function over the symmetric group in terms of characters is

$$\delta(\sigma) = \sum_\Lambda \frac{d_\Lambda}{n!} \chi_\Lambda(\sigma)\ . \tag{2.33}$$

Putting these together we find

$$\hat{\mathcal{O}}^{\mu}(\sigma) = \sum_\Lambda \sum_{i,k} \sum_\beta d_\Lambda D^{\Lambda}_{ik}(\sigma) B_{k\beta} \hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}\ . \tag{2.34}$$

We know that $\hat{\mathcal{O}}^{\mu}(\sigma)$ span the vector space of all covariant operators and they can be given as linear combinations of the operators $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ according to (2.34). Furthermore we know that the $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ have the correct counting, and so we have shown that they provide a basis for the space of all covariant operators.

2.5 $\mathrm{U}(M) \times S_n$ transformations

In this section we show that the index i in $\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta}$ transforms according to irrep Λ of the symmetric group, and that the index β (together with μ) transform according to the irrep Λ of $\mathrm{U}(M)$. For the reader who is prepared to accept this, we recommend skipping ahead to section 3. To demonstrate this we need to show that the S_n action is a left-action on σ and $\mathrm{U}(M)$ is a right-action.

2.5.1 S_n action

The action of $\tau \in S_n$ as defined in (2.3) is

$$\hat{\mathcal{O}}^{\mu}(\sigma) \to \hat{\mathcal{O}}^{\mu}(\tau\sigma) \tag{2.35}$$

Thus

$$\hat{\mathcal{O}}^{\Lambda\mu}_{ij} \to \sum_{\sigma} D^{\Lambda}_{ij}(\sigma)\hat{\mathcal{O}}^{\mu}(\tau\sigma) = \sum_{\rho} D^{\Lambda}_{ij}(\tau^{-1}\rho)\hat{\mathcal{O}}^{\mu}(\rho) = D^{\Lambda}_{ik}(\tau^{-1})\hat{\mathcal{O}}^{\Lambda\mu}_{kj} \tag{2.36}$$

It only acts on the i index.

2.5.2 $\mathrm{U}(M)$ action

In order to describe the action of $\mathrm{U}(M)$ it is convenient to introduce the following operator,

$$\hat{\mathcal{O}}^{\Lambda}_{ij;a_1\ldots a_n} = \sum_{\sigma} D^{\Lambda}_{ij}(\sigma)\hat{\mathcal{O}}_{\sigma;a_1\ldots a_n} \tag{2.37}$$

where a_i are $\mathrm{U}(M)$ indices and

$$\hat{\mathcal{O}}_{\sigma;a_1\ldots a_n} = \sigma \;\cdot\; X_{a_1} \otimes X_{a_2} \cdots \otimes X_{a_n} \;\cdot\; \sigma^{-1} \tag{2.38}$$

Under a $\mathrm{U}(M)$ transformation $X_a \to \sum_{a'=1}^{M} g_a{}^{a'} X_{a'}$ and so

$$\hat{\mathcal{O}}^{\Lambda}_{ij;\vec{a}} \to \sum_{\vec{a}} g_{\vec{a}}{}^{\vec{a}'}\hat{\mathcal{O}}^{\Lambda}_{ij;\vec{a}'} \tag{2.39}$$

where $\vec{a}$ is short-hand for $a_1 \ldots a_n$ and $g_{\vec{a}}{}^{\vec{a}'}$ stands for $g_{a_1}{}^{a'_1} \ldots g_{a_n}{}^{a'_n}$. This is the group action, if instead we consider the the action of the Lie algebra then we have $\delta X_a = \sum_{a'=1}^{M} g_a{}^{a'} X_{a'}$ and $\delta\hat{\mathcal{O}}^{\Lambda}_{ij;\vec{a}} = \sum_{\vec{a}} g_{\vec{a}}{}^{\vec{a}'}\hat{\mathcal{O}}^{\Lambda}_{ij;\vec{a}'}$,, where $g_{\vec{a}}{}^{\vec{a}'} = \sum_{\alpha=1}^{n} \delta^{a'_1}_{a_1} \ldots \delta^{a'_{\alpha-1}}_{a_{\alpha-1}} g^{a'_\alpha}_{a_\alpha} \delta^{a'_{\alpha+1}}_{a_{\alpha+1}} \ldots \delta^{a'_n}_{a_n}$, but everything else follows.

As we saw in (2.7) the operators $\hat{\mathcal{O}}(\vec{a})$ can be written as $\hat{\mathcal{O}}^{\mu}(\sigma)$. The sum over all possible values for $\vec{a}$ can be written as

$$\sum_{\vec{a}} \hat{\mathcal{O}}(\vec{a}) = \sum_{\mu}\sum_{\sigma} \frac{1}{|H_\mu|}\hat{\mathcal{O}}^{\mu}(\sigma) \tag{2.40}$$

where the division by $|H_\mu|$ accounts for the fact that $\vec{a}$ determines a unique μ but not a unique σ.

So under a $\mathrm{U}(M)$ transformation we have

$$
\begin{aligned}
\hat{\mathcal{O}}^{\Lambda\mu}_{ij} \to & \sum_{\vec{a}} g_\mu{}^{\vec{a}'} \hat{\mathcal{O}}^{\Lambda}_{ij;\vec{a}'} && (2.41)\\
= & \sum_{\mu'} \sum_{\sigma'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} \sum_{\sigma} D^\Lambda_{ij}(\sigma) \hat{\mathcal{O}}^{\mu'}_{\sigma\sigma'} && (2.42)\\
= & \sum_{\mu'} \sum_{\sigma'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} \sum_{\sigma} D^\Lambda_{ik}(\sigma) \hat{\mathcal{O}}^{\mu'}_{\sigma}\, D^\Lambda_{jk}(\sigma') && (2.43)\\
= & \sum_{\mu'} \sum_{\sigma'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} D^\Lambda_{jk}(\sigma') \hat{\mathcal{O}}^{\Lambda\mu'}_{ik} && (2.44)
\end{aligned}
$$

and we see that the $\mathrm{U}(M)$ transformation leaves the Λ and i unchanged. The linearly independent operators $\mathcal{O}^{\Lambda\mu}_{i\beta} := \sum_j \mathcal{O}^{\Lambda\mu}_{ij} B_{j\beta}$ then transform as

$$
\begin{aligned}
\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta} \to & \sum_{\mu'} \sum_{\sigma'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} \sum_{j} B_{j\beta} D^\Lambda_{jk}(\sigma') \hat{\mathcal{O}}^{\Lambda\mu'}_{ik} && (2.45)\\
= & \sum_{\mu'} \sum_{\sigma'\in S_n} \sum_{j,k} \sum_{\beta'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} B_{j\beta} D^\Lambda_{jk}(\sigma') B_{k\beta'} \hat{\mathcal{O}}^{\Lambda\mu'}_{i\beta'} && (2.46)
\end{aligned}
$$

Here to obtain the second line we have used (2.31).

To summarise we can say that under a $\mathrm{U}(M)$ transformation

$$
\mathcal{O}^{\Lambda\mu}_{i\beta} \to \sum_{\mu'} \sum_{\beta'} g_{\mu\beta}{}^{\mu'\beta'} \mathcal{O}^{\Lambda\mu'}_{i\beta'} \tag{2.47}
$$

where

$$
g_{\mu\beta}{}^{\mu'\beta'} = \sum_{\sigma'\in S_n} \sum_{j,k} \sum_{\beta'} \frac{1}{|H_{\mu'}|} g_\mu{}^{\sigma'\mu'} B_{j\beta} D^\Lambda_{jk}(\sigma') B_{k\beta'} \tag{2.48}
$$

Again the Lie algebra follows precisely from this by making g infinitesimal.

2.5.3 $\mathrm{U}(M)$ highest weight

The highest weight state (HWS) of the $\mathrm{U}(M)$ representation has a particularly simple form. The HWS has $\mu = \Lambda$ and since $g(\Lambda;\Lambda) = 1$, β takes only one value. Therefore the projector $D_{ij}(\Gamma)$ projects onto a one-dimensional subspace. We can always choose representing matrices such that

$$
D_{ij}(\Gamma) = \delta_{1i}\delta_{1j} \tag{2.49}
$$

giving the branching coefficients $B_{j\beta} = \delta_{1j}$. Hence the basis of highest weight states is given by the operators

$$
\hat{\mathcal{O}}^{\Lambda,HWS}_{i} := \hat{\mathcal{O}}^{\Lambda\Lambda}_{i\beta} = \hat{\mathcal{O}}^{\Lambda\Lambda}_{i1} = \sum_{\sigma} D^\Lambda_{i1}(\sigma) \sigma\, \mathbf{X}^\Lambda\, \sigma^{-1} \tag{2.50}
$$

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and all operators are obtained by applying lowering operators to these (the precise action of $\mathrm{U}(M)$ on the operators is given in section 2.5.2).

It is common to give irreducible representations of $\mathrm{U}(M)$ by using Young projectors acting on the tensor product of fundamental representations. This procedure is closely related to our method - in particular the highest weight state will be similar to (2.50) with D^Λ_{i1} replaced by the Young projector P^Λ_i - but does not lead to a diagonal two-point function.

3. Gauge-invariant operators

We will now construct the corresponding basis for gauge-invariant operators made from multi-traces of multi-fields. As before, it is instructive first to consider the counting of such operators.

3.1 Counting

The formula for the number $N(\mu_1,\dots\mu_M)$ of gauge-invariants operators (including multi-traces) made from fields μ_1 of X_1, μ_2 of X_2, $\dots\mu_M$ of X_M at large N is given via Pólya theory to be the coefficient of $x_1^{\mu_1}\cdots x_M^{\mu_M}$ in the partition function

$$\mathcal{Z}_{\mathrm{U}(N\to\infty)}(x_i) = \prod_{k=1}^{\infty} \frac{1}{1-(x_1^k+\cdots+x_M^k)} = \sum_\mu N(\mu_1,\dots\mu_M)x_1^{\mu_1}\cdots x_M^{\mu_M} \tag{3.1}$$

This coefficient is

$$N(\mu_1,\dots\mu_M) = \sum_R\sum_\Lambda C(R,R,\Lambda)g(\mu;\Lambda) \tag{3.2}$$

R is a representation of S_n and Λ is a representation of $\mathrm{U}(M)$ and S_n, where $n=\sum_k\mu_k$. $C(R,R,\Lambda)$ is the coefficient of Λ in the (inner) tensor product $R\otimes R$ of S_n representations. In this formula the Littlewood Richardson coefficient $g(\mu;\Lambda)$, see equation (2.11), is counting states in the $\mathrm{U}(M)$ representation Λ with field content μ and $\sum_R C(R,R,\Lambda)$ is counting the number of gauge-invariant multiplets for Λ.

This formula is discussed in [10] for the finite N case (developing the large N results of [11, 12]), where the Clebsch-multiplicities appear in properties of symmetric polynomials, from the free field partition function. We give a short proof here of the simpler large N case.

$C(R,S,T)$ is given by

$$C(R,S,T) = \sum_{C_{\mathbf{i}}\in S_n}\frac{1}{|\mathrm{Sym}(C_{\mathbf{i}})|}\chi_R(C_{\mathbf{i}})\chi_S(C_{\mathbf{i}})\chi_T(C_{\mathbf{i}}) \tag{3.3}$$

$C_{\mathbf{i}}$ represents the conjugacy class of S_n with i_1 1-cycles, i_2 2-cycles, $\dots i_n$ n-cycles. $|\mathrm{Sym}(C_{\mathbf{i}})|$ is the size of the symmetry group of the conjugacy class $C_{\mathbf{i}}$. We can immediately simplify $\sum_R C(R,R,\Lambda)$ using the orthogonality relation

$$\sum_R \chi_R(\rho)\chi_R(\tau) = |\mathrm{Sym}(\tau)|\;\;\delta([\tau]=[\rho]) \tag{3.4}$$

to get

$$\sum_R C(R,R,\Lambda) = \sum_{C_{\mathbf{i}}\in S_n} \chi_\Lambda(C_{\mathbf{i}}) \tag{3.5}$$

$N(\mu_1,\dots\mu_M)$ is the coefficient of $x_1^{\mu_1}\cdots x_M^{\mu_M}$ in

$$\prod_{k=1}^{\infty} \frac{1}{1-(x_1^k+\cdots+x_M^k)} = \sum_{i_1,i_2,\dots} (x_1+\cdots+x_M)^{i_1}\,(x_1^2+\cdots+x_M^2)^{i_2}\cdots \tag{3.6}$$

For any symmetric polynomial P the coefficient of $x_1^{\mu_1}\cdots x_M^{\mu_M}$ is given by

$$[P]_\mu = \sum_\Lambda g(\mu;\Lambda)\,[\Delta\cdot P]_{[\Lambda_1+M-1,\Lambda_2+M-2,\dots\Lambda_M]} \tag{3.7}$$

Δ is the discriminant $\Delta(x)=\prod_{i<j}(x_i-x_j)$. This formula is given in section 4.3 of Fulton and Harris [7]. Thus

$$\begin{aligned} N(\mu_1,\dots\mu_M) &= \sum_{i_1,i_2,\dots}\sum_\Lambda g(\mu;\Lambda)\left[\Delta\cdot\prod_j (x_1^j+\cdots x_M^j)^{i_j}\right]_{[\Lambda_1+M-1,\Lambda_2+M-2,\dots\Lambda_M]} \\ &= \sum_{i_1,i_2,\dots}\sum_\Lambda g(\mu;\Lambda)\chi_\Lambda(C_{\mathbf{i}}) \\ &= \sum_R\sum_\Lambda C(R,R,\Lambda)g(\mu;\Lambda) \end{aligned} \tag{3.8}$$

In the second line we have identified the Frobenius character formula [7]. Note that the quantity summed here can also be recognised as the character of the representation of S_n induced from the trivial representation of the subgroup $S_{\mu_1}\times\cdots S_{\mu_M}$

$$\psi_\Lambda(C_{\mathbf{i}}) = \sum_\Lambda g(\mu;\Lambda)\chi_\Lambda(C_{\mathbf{i}}) \tag{3.9}$$

A more explicit proof is given in appendix C.1.

3.1.1 Finite N counting

Once we interpret R as a representation of $\mathrm{U}(N)$ the counting formula

$$N(\mu_1,\dots\mu_M) = \sum_R\sum_\Lambda C(R,R,\Lambda)g(\mu;\Lambda) \tag{3.10}$$

extends to finite N, where we truncate the sum over R to Young diagrams with at most N rows. This formula was derived for the finite N case in [10].

If we decompose the partition function into $\mathrm{U}(M)$ characters

$$\chi^\Lambda(x_i) = \sum_\mu g(\mu;\Lambda)x_1^{\mu_1}x_2^{\mu_2}\dots x_M^{\mu_M} \tag{3.11}$$

then we get

$$\mathcal{Z}_{\mathrm{U}(N)}(x_i) = \sum_R\sum_\Lambda C(R,R,\Lambda)\chi^\Lambda(x_i) \tag{3.12}$$

where the coefficient of each character χ^Λ in the partition function gives the number of $\mathrm{U}(M)$ multiplets Λ in the theory.

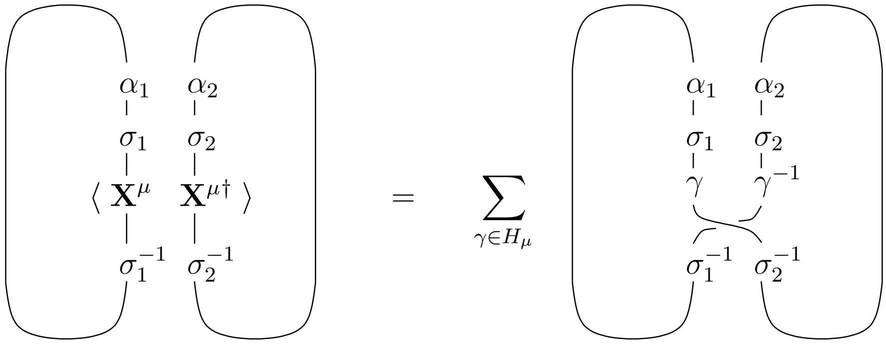

Figure 2: Two-point function for the gauge-invariant operator.

3.2 Operators and correlators

3.2.1 The trace basis

We start by defining multitraces $\mathcal{O}^\mu(\alpha,\sigma)$ of the basic gauge-covariant operator $\mathcal{O}^\mu(\sigma)$ where the permutation alpha determines the cycles of the trace

$$\mathcal{O}^\mu(\alpha,\sigma) = \mathrm{tr}(\alpha\hat{\mathcal{O}}^\mu(\sigma)) = \mathrm{tr}(\ \alpha\ \sigma\ X_1^{\mu_1} \otimes \cdots \otimes X_M^{\mu_M}\ \sigma^{-1}) = tr(\alpha\ \sigma\ \mathbf{X}^\mu\ \sigma^{-1}) \tag{3.13}$$

The trace is being taken in $V^{\otimes n}$. Using permutations α to parametrize multi-traces is useful and non-trivial already in the half-BPS case, where a single matrix is involved [1]. $\sigma \in S_n$ controls the order of the fields and ties in closely with the $\mathrm{U}(M)$ representation of the operator.

This operator has two symmetries that leave it unchanged

$$\begin{aligned}
(\alpha,\sigma) &\to (\pi\alpha\pi^{-1},\pi\sigma) && \pi \in S_n \\
(\alpha,\sigma) &\to (\alpha,\sigma\gamma) && \gamma \in H_\mu = S_{\mu_1} \times \cdots \times S_{\mu_M} \subset S_n
\end{aligned}$$

With the π symmetry we can remove the σ dependence of our operator. If we then take equivalence classes $[\alpha]$ of the equivalence relation

$$\alpha \sim \gamma\alpha\gamma^{-1} \tag{3.14}$$

for $\gamma \in H_\mu$ then $\mathrm{tr}([\alpha]\ \mathbf{X}^\mu)$ is the independent *trace basis*. It has the correct counting of gauge-invariant operators for fixed μ field content (see section C.3).

The two-point function of $\mathcal{O}^\mu(\alpha,\sigma)$ is given by

$$\left\langle \mathcal{O}^{\mu^{(1)}}(\alpha_1,\sigma_1)\ \mathcal{O}^{\dagger\mu^{(2)}}(\alpha_2,\sigma_2) \right\rangle = \delta^{\mu^{(1)}\mu^{(2)}} \sum_{\gamma \in H_{\mu^{(1)}}} \mathrm{tr}(\alpha_1\sigma_1\gamma\sigma_2^{-1}\alpha_2\sigma_2\gamma^{-1}\sigma_1^{-1}) \tag{3.15}$$

This is easily derived from the diagrammatic presentation of correlators as in figure 2. Such diagrammatic manipulations are also useful in deriving properties of half-BPS correlators [13].

3.2.2 A diagonal basis

First we Fourier transform the σ and multiply by the branching coefficient as we did for the covariant case (from here on we will use the Einstein summation convention)

$$\mathcal{O}^{\Lambda\mu}_{i\beta}(\alpha) = \frac{1}{n!}\sum_{\sigma} B_{j\beta} D^{\Lambda}_{ij}(\sigma)\,\mathrm{tr}(\alpha\sigma\;\mathbf{X}^{\mu}\;\sigma^{-1}) \tag{3.16}$$

$D^{\Lambda}_{ij}(\sigma)$ is the orthogonal representing matrix. This operator is the trace with α of the covariant operator (2.24): $\mathcal{O}^{\Lambda\mu}_{i\beta}(\alpha) = \mathrm{tr}(\alpha\;\hat{\mathcal{O}}^{\Lambda\mu}_{i\beta})$. The two point function follows from the covariant result (2.26)

$$\left\langle \mathcal{O}^{\Lambda_1\mu^{(1)}}_{i_1\beta_1}(\alpha_1)\mathcal{O}^{\dagger\;\Lambda_2\mu^{(2)}}_{i_2\beta_2}(\alpha_2)\right\rangle = \delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta_{\beta_1\beta_2}\frac{|H_\mu|}{n!d_{\Lambda_1}}\sum_{\rho} D^{\Lambda_1}_{i_1i_2}(\rho)\,\mathrm{tr}(\alpha_1\rho\alpha_2\rho^{-1}) \tag{3.17}$$

The trace is being taken in $V^{\otimes n}$ and we have Schur-Weyl duality

$$V^{\otimes n} = \oplus_T V^{\mathrm{U}(N)}_T \otimes V^{S_n}_T \tag{3.18}$$

So we can use $\mathrm{tr}(\sigma\;\mathbb{I}) = \sum_T \chi_T(\sigma)\chi_T(\mathbb{I}) = \sum_T \chi_T(\sigma)\mathrm{Dim}T$, where $\mathrm{Dim}T$ is the dimension of the $\mathrm{U}(N)$ representation T, to get

$$\delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta_{\beta_1\beta_2}\frac{|H_{\mu^{(1)}}|}{n!d_{\Lambda_1}}\sum_{\rho} D^{\Lambda_1}_{i_1i_2}(\rho)\sum_{T}\mathrm{Dim}T\;D^T_{ab}(\alpha_1)D^T_{bc}(\rho)D^T_{cd}(\alpha_2)D^T_{da}(\rho^{-1}) \tag{3.19}$$

Now use the orthogonality of the matrices and identity (A.9) on the ρ sum

$$\delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta_{\beta_1\beta_2}\frac{|H_{\mu^{(1)}}|}{d^2_{\Lambda_1}}\sum_{T}\mathrm{Dim}T\;D^T_{ab}(\alpha_1)D^T_{cd}(\alpha_2)\sum_{\tau} S^{\tau,\Lambda_1\;T\;T}_{\;\;i_1\;\;a\;b}\,S^{\tau,\Lambda_1\;T\;T}_{\;\;i_2\;\;c\;d} \tag{3.20}$$

The S here are Clebsch-Gordan coefficients for the (inner) tensor product of S_n. section A.1 explains where these come from in more detail.

Next we Fourier transform α in the same way to get the operator

$$\mathcal{O}^{\Lambda\mu R}_{i\beta;\,kl} = \frac{1}{n!}\sum_{\alpha} D^R_{kl}(\alpha)\frac{1}{n!}\sum_{\sigma} B_{j\beta}D^{\Lambda}_{ij}(\sigma)\,\mathrm{tr}(\alpha\sigma\;\mathbf{X}^{\mu}\;\sigma^{-1}) \tag{3.21}$$

If we apply this sum over α_1 in (3.20) then we can factor out

$$\sum_{\alpha_1} D^{R_1}_{k_1l_1}(\alpha_1)D^T_{ab}(\alpha_1) = \frac{n!}{d_{R_1}}\delta^{R_1T}\delta_{k_1a}\delta_{l_1b} \tag{3.22}$$

We have used (A.3). Thus we get

$$\left\langle \mathcal{O}^{\Lambda_1\mu^{(1)}R_1}_{i_1\beta_1;\;\;k_1l_1}\mathcal{O}^{\dagger\Lambda_2\mu^{(2)}R_2}_{i_2\beta_2;\;\;k_2l_2}\right\rangle = \delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta_{\beta_1\beta_2}\delta^{R_1R_2}\times \tag{3.23}$$
$$\times\frac{|H_{\mu^{(1)}}|\mathrm{Dim}R_1}{d^2_{\Lambda_1}d^2_{R_1}}\sum_{\tau} S^{\tau,\Lambda_1\;R_1\;R_1}_{\;\;i_1\;\;k_1\;\;l_1}\,S^{\tau,\Lambda_1\;R_1\;R_1}_{\;\;i_2\;\;k_2\;\;l_2}$$

Finally we define

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = S^{\tau,\Lambda}{}_{i}{}^{R}_{k}{}^{R}_{l}\; \mathcal{O}^{\Lambda\mu R}_{i\beta;\,kl} \tag{3.24}$$

Note that β runs over 1 to $g(\mu;\Lambda)$ and τ runs over 1 to $C(R,R;\Lambda)$ which are the factors appearing in the counting (3.2). Some of these operators are worked out in appendix E.

Using (3.23) and the orthogonality of the Clebsch-Gordan coefficients (A.6)

$$\left\langle \mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1} \mathcal{O}^{\dagger\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2} \right\rangle = \delta^{\mu^{(1)}\mu^{(2)}} \delta^{\Lambda_1\Lambda_2} \delta_{\beta_1\beta_2} \delta^{R_1R_2} \delta_{\tau_1\tau_2} \frac{|H_{\mu^{(1)}}|\mathrm{Dim}R_1}{d_{R_1}^2} \tag{3.25}$$

3.3 Simplified operator

The operator $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ can be written more simply in the trace basis, removing the σ redundancy. Take the operator

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{(n!)^2} \sum_{\sigma,\alpha} B_{j\beta}\, S^{\tau,\Lambda}{}_{i}{}^{R}_{k}{}^{R}_{l}\; D^{\Lambda}_{ij}(\sigma) D^{R}_{kl}(\alpha)\, \mathrm{tr}(\alpha\sigma\, \mathbf{X}^{\mu}\, \sigma^{-1}) \tag{3.26}$$

and apply identity (A.8) to $S^{\tau,\Lambda}{}_{i}{}^{R}_{k}{}^{R}_{l}\; D^{\Lambda}_{ij}(\sigma)$ to get

$$\begin{aligned} \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} &= \frac{1}{(n!)^2} \sum_{\sigma,\alpha} B_{j\beta}\, S^{\tau,\Lambda}{}_{j}{}^{R}_{p}{}^{R}_{q}\; D^{R}_{kp}(\sigma) D^{R}_{lq}(\sigma) D^{R}_{kl}(\alpha)\, \mathrm{tr}(\alpha\sigma\, \mathbf{X}^{\mu}\, \sigma^{-1}) \\ &= \frac{1}{(n!)^2} \sum_{\sigma,\alpha} B_{j\beta}\, S^{\tau,\Lambda}{}_{j}{}^{R}_{p}{}^{R}_{q}\; D^{R}_{pq}(\sigma^{-1}\alpha\sigma)\, \mathrm{tr}(\sigma^{-1}\alpha\sigma\, \mathbf{X}^{\mu}) \end{aligned} \tag{3.27}$$

In the second line we have exploited the fact that we are using orthogonal representing matrices. Finally we make a substitution of summation variables and do the sum over σ

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{n!} \sum_{\alpha} B_{j\beta}\, S^{\tau,\Lambda}{}_{j}{}^{R}_{p}{}^{R}_{q}\; D^{R}_{pq}(\alpha)\, \mathrm{tr}(\alpha\, \mathbf{X}^{\mu}) \tag{3.28}$$

It is easy to check that this operator is constant on the equivalence classes of α given by (3.14).

3.4 Half-BPS case

The Schur polynomial diagonalisation of [1] is a special case of this operator, when the representation is the trivial one $\Lambda = [n]$ and we consider the highest weight state of this representation $\mu_1 = n$.

In this case the counting is

$$N(n) = \sum_R \sum_\Lambda C(R,R,\Lambda) g([n];\Lambda) = \sum_R C(R,R,[n]) = \sum_R 1 = p(n) \tag{3.29}$$

$p(n)$ is the number of partitions of n and we have used the fact that for the trivial representation $C(R,R,[n]) = 1$, since $R \otimes [n] = R$. Also $g([n];\Lambda)$ is only non-zero for the trivial representation $\Lambda = [n]$, when it is one.

For the Clebsch-Gordan coefficient we have

$$S^{[n]}_{1}{}^{R}_{k}{}^{R}_{l} = \frac{1}{\sqrt{d_R}} \delta_{kl} \tag{3.30}$$

This follows from setting T to be the trivial representation in (A.9) and then using the orthogonality relation (A.3). The operator is therefore

$$\mathcal{O}^{[n],R} = \frac{1}{\sqrt{d_R}} \mathcal{O}^{\Lambda}_{11}{}^{R}_{kk} = \frac{1}{\sqrt{d_R}} \chi_R(X_1) \tag{3.31}$$

3.5 Basis from projection

All gauge invariant operators can be written as linear combinations of operators of the form $\mathcal{O}^\mu(\alpha) = \mathrm{tr}(\alpha X_1^{\otimes \mu_1} \otimes \cdots \otimes X_M^{\otimes \mu_M})$. In order to obtain a basis of operators with the correct counting we need to mod out by the symmetry

$$\mathcal{O}^\mu(\alpha) = \mathcal{O}^\mu(\gamma^{-1}\alpha\gamma) \qquad \forall \gamma \in H_\mu \tag{3.32}$$

Taking the Fourier transform we wish to find a basis for the space of operators $\mathcal{O}^\mu{}^R_{kl} := \frac{1}{n!}\sum_\alpha D^R_{kl}(\alpha)\mathcal{O}^\mu(\alpha)$ up to the symmmetry

$$\mathcal{O}^\mu{}^R_{kl} = D^R_{km}(\gamma) D^R_{ln}(\gamma) \mathcal{O}^\mu{}^R_{mn} \qquad \forall \gamma \in H_\mu \tag{3.33}$$

which follows directly from (3.32).

Now $\mathcal{O}^\mu{}^R_{kl}$ lies in the tensor product of S_n irreps. $R \otimes R$ (carried by the two indices k, l) and the problem becomes that of finding the subspace of this which is invariant under H_μ. The projector which projects onto this space is found by taking the sum of all the permutations in H_μ, so we are seeking the set of operators $\mathcal{O}^\mu{}^R_{kl}$ for which

$$\mathcal{O}^\mu{}^R_{kl} = P_{kl;mn} \mathcal{O}^\mu{}^R_{mn} \tag{3.34}$$

$$P_{kl;mn} := \frac{1}{|H_\mu|} \sum_{\gamma \in H_\mu} D^R_{km}(\gamma) D^R_{ln}(\gamma) \tag{3.35}$$

If we write $D^R_{km}(\gamma) D^R_{ln}(\gamma)$ as a matrix in $R \otimes R$ we can decompose it by inserting complete sets of states

$$\begin{aligned} \frac{1}{|H_\mu|} \sum_{\gamma \in H_\mu} D^R_{km}(\gamma) D^R_{ln}(\gamma) &= \langle R, k; R, l | \, \Gamma \, | R, m; R, n \rangle \\ &= \sum_{\Lambda, \tau; \Lambda', \tau'} \langle R, k; R, l | \Lambda, \tau, s \rangle \langle \Lambda, \tau, s | \, \Gamma \, | \Lambda', \tau', t \rangle \langle \Lambda', \tau', t | R, m; R, n \rangle \\ &= \sum_{\Lambda, \tau} \langle R, k; R, l | \Lambda, \tau, s \rangle \langle \Lambda, \tau, s | \, \Gamma \, | \Lambda, \tau, t \rangle \langle \Lambda, \tau, t | R, m; R, n \rangle \end{aligned} \tag{3.36}$$

where $\Gamma = \sum_{\gamma \in H_\mu} \gamma / |H_\mu|$. The factors $\langle R, k; R, l | \Lambda, \tau, s \rangle$ are just the Clebsch-Gordan coefficients (A.5), so we can write the projector

$$P_{kl;mn} = \sum_\Lambda \sum_\tau \sum_{s,t} D^\Lambda_{st}(\Gamma) S^{\tau, \Lambda}{}_{s}{}^{R}_{k}{}^{R}_{l} S^{\tau, \Lambda}{}_{t}{}^{R}_{m}{}^{R}_{n} \tag{3.37}$$

We then write $D^\Lambda_{ts}(\Gamma) = B_{t\beta} B_{s\beta}$ as in (2.20) and so we can decompose the projector into the form (B.2)

$$P_{kl;mn} = \sum_{\Lambda, \tau, \beta} b_{kl;\Lambda\tau\beta} b_{mn;\Lambda\tau\beta} \tag{3.38}$$

where

$$b_{kl;\Lambda\tau\beta} := B_{s\beta} S^{\tau,\Lambda}{}_{s}{}^{R}{}_{k}{}^{R}{}_{l} \tag{3.39}$$

Now we have precisely the situation described in appendix B, with the relation (B.3) following from (A.6). Therefore

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} := b_{kl;\Lambda\tau\beta}\mathcal{O}^{\mu}{}^{R}_{kl} = B_{s\beta} S^{\tau,\Lambda}{}_{s}{}^{R}{}_{k}{}^{R}{}_{l}\mathcal{O}^{\mu}{}^{R}_{kl} \tag{3.40}$$

provides a basis of gauge invariant operators.

Notice that here we started by ignoring the the $\mathrm{U}(M)$ aspect of the operators completely and yet we find the $\mathrm{U}(M)$ representation Λ appearing as before.

3.6 Recovering the trace basis

It will be useful to invert the map so that we can write $\mathcal{O}^{\mu}(\alpha)$ in terms of $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$. Given that $\mathcal{O}^{\mu}(\alpha) = \mathcal{O}^{\mu}(\gamma\alpha\gamma^{-1})$, for the operator $\mathcal{O}^{\mu}{}^{R}_{kl} = \frac{1}{n!}\sum_{\alpha} D^{R}_{kl}(\alpha)\mathcal{O}^{\mu}(\alpha)$ we have

$$\begin{aligned} \mathcal{O}^{\mu}{}^{R}_{kl} &= \frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu} D^{R}_{km}(\gamma) D^{R}_{ln}(\gamma)\mathcal{O}^{\mu}{}^{R}_{mn} \\ &= P_{kl;mn}\mathcal{O}^{\mu}{}^{R}_{mn} \\ &= \sum_{\Lambda,\tau,\beta} b_{kl;\Lambda\tau\beta}\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \end{aligned} \tag{3.41}$$

where we have used the projector identities in the previous section.

Now use

$$\sum_R d_R D^{R}_{kl}(\alpha') D^{R}_{kl}(\alpha) = \sum_R d_R \chi_R(\alpha'\alpha^{-1}) = n!\delta(\sigma'\sigma^{-1}) \tag{3.42}$$

to get

$$\mathcal{O}^{\mu}(\alpha) = \mathrm{tr}(\alpha\,\mathbf{X}^{\mu}) = \sum_R d_R D^{R}_{kl}(\alpha) B_{s\beta} S^{\tau,\Lambda}{}_{s}{}^{R}{}_{k}{}^{R}{}_{l}\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \tag{3.43}$$

We know that the $\mathcal{O}^{\mu}(\alpha)$ span the vector space of all gauge-invariant operators and they can be given as linear combinations of the operators $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$. Since the $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ also have the correct counting, they provide a basis.

4. Including fermions

It is interesting to extend these results from the case of Lie groups $\mathrm{U}(M)$ to super Lie groups $\mathrm{U}(M_1|M_2)$. Indeed the space of eighth BPS operators in $\mathcal{N}$=4 SYM corresponds to the case $\mathrm{U}(3|2)$, the three scalar fields X, Y, Z of the $\mathrm{U}(3)$ sector combining with two fermions, $\bar{\lambda}^{1}{}_{2}, \bar{\lambda}^{1}{}_{1}$ (in the notation of [14]). The adjoint of the fermions $\bar{\lambda}^{1}{}_{\dot{a}}$ is denoted λ_{1a}. The two-point function of the two fermionic fields is then given by

$$\left\langle (\bar{\lambda}^{1}{}_{\dot{a}})^{i}_{j}(\lambda_{1a})^{k}_{l}\right\rangle = \delta_{\dot{a}a}\,\delta^{i}_{l}\delta^{k}_{j} \tag{4.1}$$

Note that here, as for the bosonic case, we have ignored the x dependence which is $(\delta_{a\dot{a}}x_{12}^0 - \sigma_{a\dot{a}}^i x_{12}^i)/x_{12}^4$. By taking a limit where separation in time x_{12}^0 dominates the separations in space x_{12}^i, we have that the two-point function is proportional to $\delta_{a\dot{a}}$

The full set of fundamental fields in the sector is thus denoted X_a as previously, but where X_a is bosonic for $a = 1 \dots M_1$ and fermionic for $a = M_1 + 1 \dots M_1 + M_2$. The main difference this makes as far as we are concerned is that we pick up an extra minus sign when two fermionic fields are swapped. So

$$(X_a)_i^j (X_b)_k^l = (-1)^{\epsilon(X_a)\epsilon(X_b)} (X_b)_k^l (X_a)_i^j \tag{4.2}$$

where we have defined the Grassmann parity of X_a as

$$\epsilon(X_a) = 0 \qquad a = 1 \dots M_1 \tag{4.3}$$

$$\epsilon(X_a) = 1 \qquad a = M_1 + 1 \dots M_1 + M_2 \tag{4.4}$$

We will also find it useful to define the Grassmann parity of permutations, given $\mathbf{X}^\mu$. We first define it for transpositions

$$\epsilon((ij)) = 0 \qquad i \text{ or j} = 1 \dots \mathrm{n}_1 \tag{4.5}$$

$$\epsilon((ij)) = 1 \qquad i \text{ and j} = \mathrm{n}_1 + 1 \dots \mathrm{n}_1 + \mathrm{n}_2 \tag{4.6}$$

and extend it to all permutations by insisting that

$$\epsilon(\sigma\tau) = \epsilon(\sigma) + \epsilon(\tau) \mod 2 \tag{4.7}$$

Here $n_1 = \sum_{k=1}^{M_1} \mu_k$ is the total number of bosonic fields, and $n_2 = \sum_{k=M_1+1}^{M_1+M_2} \mu_k$, the total number of fermionic fields with $n = n_1 + n_2$. Then the covariant two point function of operators $\mathbf{X}^\mu$ (cf. (2.14) for the bosonic case) contains the following minus sign

$$\langle (\mathbf{X}^\mu)_J^I (\mathbf{X}^{\dagger\mu})_L^K \rangle = (-1)^{\epsilon(\gamma)} (\gamma)_L^I (\gamma^{-1})_J^K \tag{4.8}$$

and this structure allows us to apply the results from the purely bosonic case with only minor modification.

Firstly consider the gauge covariant operators. We will proceed in precise analogy to the bosonic case. We may define operators precisely as in (2.16)

$$\hat{\mathcal{O}}_{ij}^{\Lambda\mu} = \frac{1}{n!} \sum_\sigma D_{ij}^\Lambda(\sigma) \hat{\mathcal{O}}^\mu(\sigma) \tag{4.9}$$

The manipulations of (2.19) then follow through as previously but with the additional minus sign associated with γ. This additional minus sign accompanying γ means that in the final line of (2.19), the two-point function, Γ becomes $\Gamma = \frac{1}{|H_\mu|} \sum_{\gamma \in H_\mu} (-1)^{\epsilon(\gamma)} \gamma$. This means that $D_{jk}^\Lambda(\Gamma)$ becomes a projector from the representation space of Λ onto the subspace which is invariant under H up to a sign, ie $D_{ij}^\Lambda(\sigma) D_{jk}^\Lambda(\Gamma) = (-1)^{\epsilon(\sigma)} D_{ik}^\Lambda(\Gamma)$. Since it is a projector this can be written in terms of branching coefficients as in (2.20). The Kostka number becomes equal to the Littlewood-Richardson coefficient for the appearance of Λ in

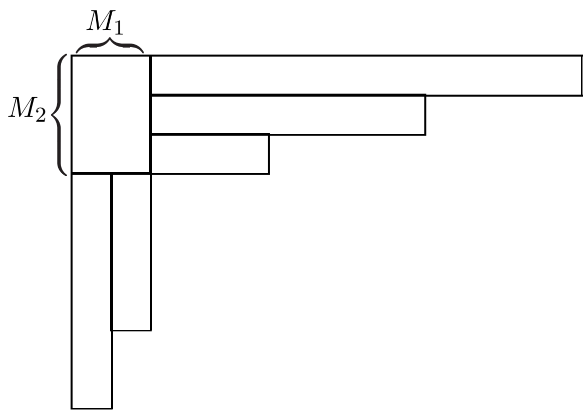

Figure 3: Allowed shape for the Young tableau of the representations Λ of $\mathrm{U}(M_1|M_2)$.

the tensor product of trivial single-row representations and antisymmetric representations $[\mu_1] \otimes \cdots \otimes [\mu_{M_1}] \otimes [1^{\mu_{M_1+1}}] \otimes \cdots \otimes [1^{\mu_{M_1+M_2}}]$. The formulae for the covariant operators should all follow through also without additional changes as they are essentially linear combinations of traces of the covariant operators already considered. In particular the operators defined in (3.24) or (3.28) are modified only in the afore-mentioned modification of the branching coefficients.

Furthermore the counting formula will be identical to (3.8) namely

$$N(\mu_1, \ldots \mu_{M_1+M_2}) = \sum_R \sum_\Lambda C(R, R, \Lambda) g(\mu; \Lambda) \tag{4.10}$$

the only difference being in the definition of the Kostka number and in the allowed representations Λ. The allowed $\mathrm{U}(M_1|M_2)$ representations Λ have the shape as shown in figure 3. The first M_1 rows are unbounded, but rather more unusually, the first M_2 columns are also unbounded. See, for example [15] for more information on representations of supergroups and supertableaux.

4.1 Single fermion

The simplest example involving fermions is given by $\mathrm{U}(M_1|M_2) = \mathrm{U}(0|1)$ corresponding to a single fermion. The allowed representations Λ are the totally antisymmetric reps, $\Lambda = [1^n]$ and the counting becomes

$$N(n) = \sum_R \sum_\Lambda C(R, R, \Lambda) g([1^n]; \Lambda) = \sum_R C(R, R, [1^n]) = \sum_{R=\tilde{R}} 1 \tag{4.11}$$

where the final sum indicates a sum over self-conjugate representations. This follows from the fact that $R \otimes [1^n] = \tilde{R}$, where $\tilde{R}$ is the partition conjugate to R, obtained by exchanging the rows and columns of R.

One can count the allowed operators for $N > n$ as follows. Single trace operators must have an odd number of fields (otherwise they vanish, for example $\mathrm{tr}(\psi\psi) = \psi_i^j \psi_j^i = -\psi_j^i \psi_i^j = 0$). Multitrace operators are then made of single-trace operators with an odd number of fields in each, with the restriction that you cannot have the same single trace term twice (otherwise it vanishes by anti-symmetry). So all our operators have the form

$$\mathrm{tr}(\psi^{2k_1+1})\,\mathrm{tr}(\psi^{2k_2+1})\ldots\mathrm{tr}(\psi^{2k_l+1}) \qquad k_1>k_2>\cdots>k_l\geq 0 \tag{4.12}$$

The map between these operators and self-conjugate Young-tableaux with k_j+j boxes in the jth row and column gives a one-to-one correspondence between multi-trace operators of a single matrix-valued fermion and self-conjugate Young tableaux.

5. Extremal higher-point correlators

We have, using the simplified form of the operators in (3.28)

$$\begin{aligned}
&\langle \mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}(x_1)\ \mathcal{O}^{\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2}(x_2)\ \mathcal{O}^{\dagger\ \Lambda_3\mu^{(3)},R_3}_{\beta_3,\tau_3}(0)\rangle\\
&\quad = x_1^{-2n_1}x_2^{-2n_2}\delta^{(\mu^{(1)}+\mu^{(2)}),\mu^{(3)}}\frac{|H_\mu|\mathrm{Dim}R_3}{d_{R_1}d_{R_2}d_{R_3}}\\
&\qquad\qquad S^{\tau_1,\Lambda_1}{}_{j_1}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1}\ S^{\tau_2,\Lambda_2}{}_{j_2}{}^{R_2}{}_{p_2}{}^{R_2}{}_{q_2}\ S^{\tau_3,\Lambda_3}{}_{j_4}{}^{R_3}{}_{p_3}{}^{R_3}{}_{q_3}\\
&\qquad\qquad\qquad D^{\Lambda_3}_{j_4j_3}(\sigma_{12})\ B_{j_3\beta_3}B_{j_1\beta_1}B_{j_2\beta_2}\ B^{R_3\to R_1\circ R_2;\beta_4}_{q_3;\ \ p_1\ \ p_2}\ B^{R_3\to R_1\circ R_2;\beta_4}_{p_3;\ \ q_1\ \ q_2}
\end{aligned} \tag{5.1}$$

All repeated indices in the r.h.s., which do not appear on the l.h.s., are summed. This is conveniently represented in a diagram as in figure 4. The diagram contains two kinds of vertices. Filled circles labelled by τ_1,τ_2,τ_3 represent inner-product couplings (or Clebsch-Gordan coefficients). Open circles represent outer-product couplings (or branching coefficients) and are labelled by $\beta_1,\beta_2\ldots$. The diagram gives a simpler (with fewer indices) and accurate representation of (the non-trivial part in the last two lines of) formula (5.1) from which the formula can be reconstructed unambiguously. To reconstruct the formula from the diagram, we insert indices for states along the edges of the diagram. We write S-factors coupling the indices incident on a filled circle and B-factors coupling the indices incident on the open circles. When an open circle has a single incident line, it stands for a branching of a symmetric group irrep into the trivial irrep of a specified sub-group determined by the μ vector associated with the corresponding operator. We could make the μ dependence explicit in these vertices of the diagram or in the equivalent $B_{j\beta}$ in the formula, but we have followed the convention of the earlier parts of the paper, of leaving that μ dependence implicit. The permutation σ_{12} re-orders the sequence of X,Y,Z coming from the two operators $\mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}$ and $\mathcal{O}^{\Lambda_2\mu^{(1)},R_2}_{\beta_2,\tau_2}$. They lead to operators $\mathrm{tr}_{V^{\otimes(n_1+n_2)}}((\alpha_1\cdot\alpha_2)\mathbf{X}^{\mu^{(1)}}\cdot\mathbf{X}^{\mu^{(2)}})$ which can be re-written as $\mathrm{tr}_{V^{\otimes(n_1+n_2)}}((\alpha_1\cdot\alpha_2)\sigma_{12}\mathbf{X}^{\mu^{(1)}+\mu^{(2)}}\sigma_{12}^{-1})$.

The product of branching coefficients is obtained from the sum over matrix elements.

$$\begin{aligned}
&\frac{d_{R_1}d_{R_2}}{n_1!n_2!}\sum_{\alpha_1,\alpha_2}D^{R_1}_{p_1q_1}(\alpha_1)D^{R_2}_{p_2q_2}(\alpha_1)D^{R_3}_{p_3q_3}(\alpha_1\circ\alpha_2)\\
&\qquad\qquad = \sum_{\beta_4}B^{R_3\to R_1\circ R_2;\beta_4}_{q_3;\ \ p_1\ \ p_2}\ B^{R_3\to R_1\circ R_2;\beta_4}_{p_3;\ \ q_1\ \ q_2}
\end{aligned} \tag{5.2}$$

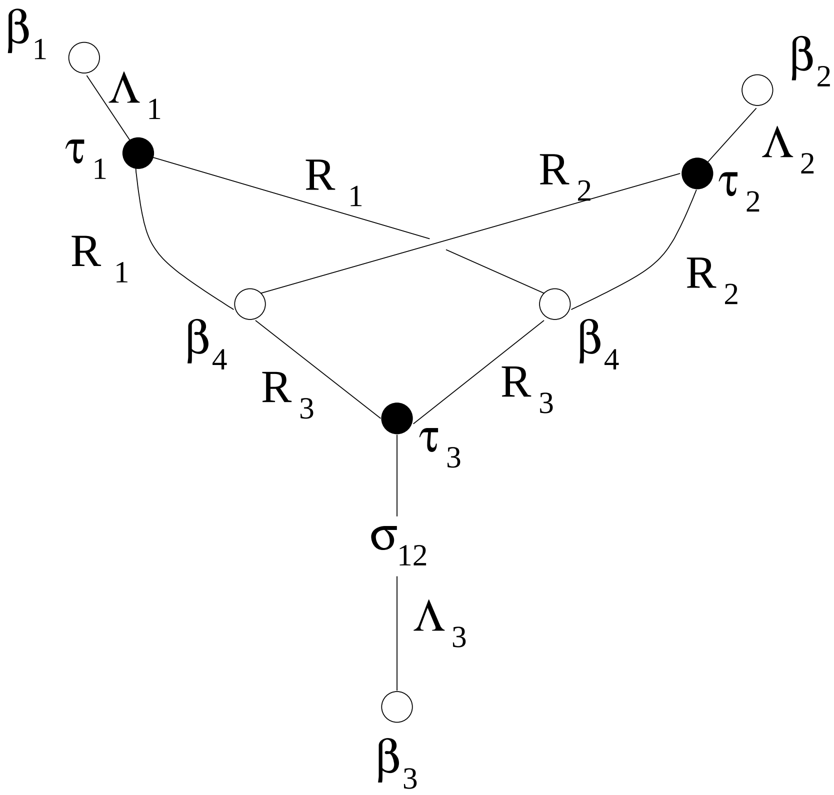

Figure 4: Diagram for three-point function

The branching coefficients express the change of basis from that of states in an irrep R_3 of S_{n_3} to that of states in an irrep $R_1 \circ R_2$ of the $S_{n_1} \times S_{n_2}$ subgroup of S_{n_3} (where $n_3 = n_1 + n_2$).

The extremal 4-point function can be written in terms of branching coefficients generalising those of (5.1), now involving 4-valent branching vertices of irreps of $S_{n_1+n_2+n_3}$ into $S_{n_1} \times S_{n_2} \times S_{n_3}$. The diagram is an easily guessed generalisation of figure 4 which is given as figure 5. Here σ_{123} is a permutation which re-orders $\mathbf{X}^{\mu^{(1)}} \cdot \mathbf{X}^{\mu^{(2)}} \cdot \mathbf{X}^{\mu^{(3)}}$ into $\mathbf{X}^{\mu^{(1)}+\mu^{(2)}+\mu^{(3)}} = \mathbf{X}^{\mu^{(4)}}$. By performing the reduction of S_n into $S_{n_1+n_2} \times S_{n_3}$ and then the reduction of $S_{n_1+n_2}$ to $S_{n_1} \times S_{n_2}$ we can split the 4-valent vertices into pairs as in figure 6.

Another approach to the 4-point function is to express the product of the first two operators in terms of 3-point functions as follows

$$\begin{aligned} &\mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}(x)\mathcal{O}^{\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2}(x) \\ &\quad = \delta^{\mu^{(1)}+\mu^{(2)},\mu} \sum_{R,\Lambda,\beta,\tau} \frac{\langle \mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}(x)\mathcal{O}^{\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2}(x)\mathcal{O}^{\dagger\ \Lambda\mu,R}_{\beta,\tau}(0)\rangle}{\langle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}(0)\mathcal{O}^{\dagger\ \Lambda\mu,R}_{\beta,\tau}(x)\rangle}\ \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}(x) \end{aligned} \tag{5.3}$$

When we write out the l.h.s. of (5.3) we have a product of traces

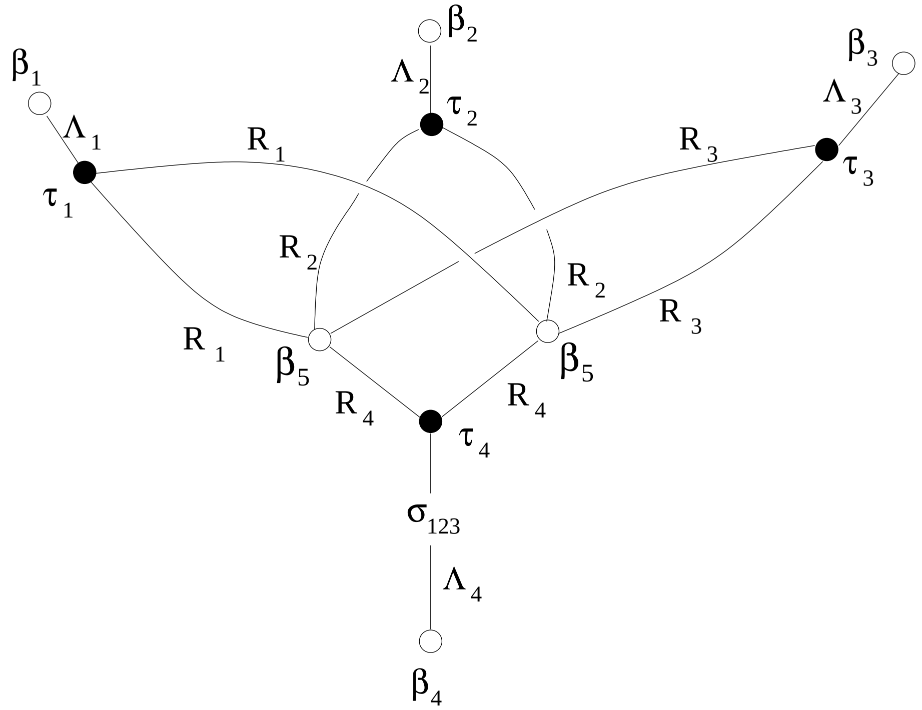

Figure 5: Diagram for four-point function

$\mathrm{tr}_{V^{\otimes n_1}}(\alpha_1 \mathbf{X}^{\mu^{(1)}})\, \mathrm{tr}_{V^{\otimes n_2}}(\alpha_2 \mathbf{X}^{\mu^{(2)}})$. This can be re-written and manipulated as follows

$$
\begin{aligned}
&\mathrm{tr}_{V^{\otimes n_1+n_2}}((\alpha_1 \cdot \alpha_2)\sigma_{12} \mathbf{X}^{\mu^{(1)}+\mu^{(2)}} \sigma_{12}^{-1}) \\
&\qquad = \sum_{R_3} \frac{d_{R_3} \chi_{R_3}(\alpha_3)}{n_3!}\, \mathrm{tr}_{V^{\otimes n_3}}(\alpha_3 (\alpha_1 \cdot \alpha_2)\sigma_{12} \mathbf{X}^{\mu^{(3)}} \sigma_{12}^{-1}) \\
&\qquad = \frac{1}{|H|} \sum_{\gamma} \sum_{R_3} \frac{d_{R_3} \chi_{R_3}(\alpha_3)}{n_3!}\, \mathrm{tr}_{V^{\otimes n_3}}(\alpha_3 (\alpha_1 \cdot \alpha_2)\sigma_{12} \gamma \mathbf{X}^{\mu^{(3)}} \gamma^{-1} \sigma_{12}^{-1})
\end{aligned} \tag{5.4}
$$

In the second line we have defined $n_3 = n_1 + n_2$, $\mu^{(3)} = \mu^{(1)} + \mu^{(2)}$ and inserted a complete set of projectors for Young diagrams in $V^{\otimes n_3}$. In the third line we have used the symmetry of $\mathbf{X}^{\mu^{(3)}}$ under conjugation by elements in H_3 to insert a group average. After a redefinition of $\gamma^{-1}\sigma_{12}^{-1}\alpha_3(\alpha_1 \cdot \alpha_2)\sigma_{12}\gamma = \tilde{\alpha}_3$ it is easy to recognise the equality (5.3).

If we derive the 4-point extremal correlator by using (5.3) twice, we get a sum over Λ, R, β, τ of figure 7. This is also equivalent to figure 5 and figure 6. All these equivalences can be seen as simple moves on the diagrams. This is reminiscent of applications of group-theoretic quantities to topology via topological field theories, where such moves are related to topological invariances of triangulations and such [16 − 18]. In this case the precise topological meaning of these correlators remains to be clarified. Perhaps relating the correlators to a topological field theory - which would have to be one based on symmetric groups (constructed along the lines of [19]) - might be a way to progress in this direction.

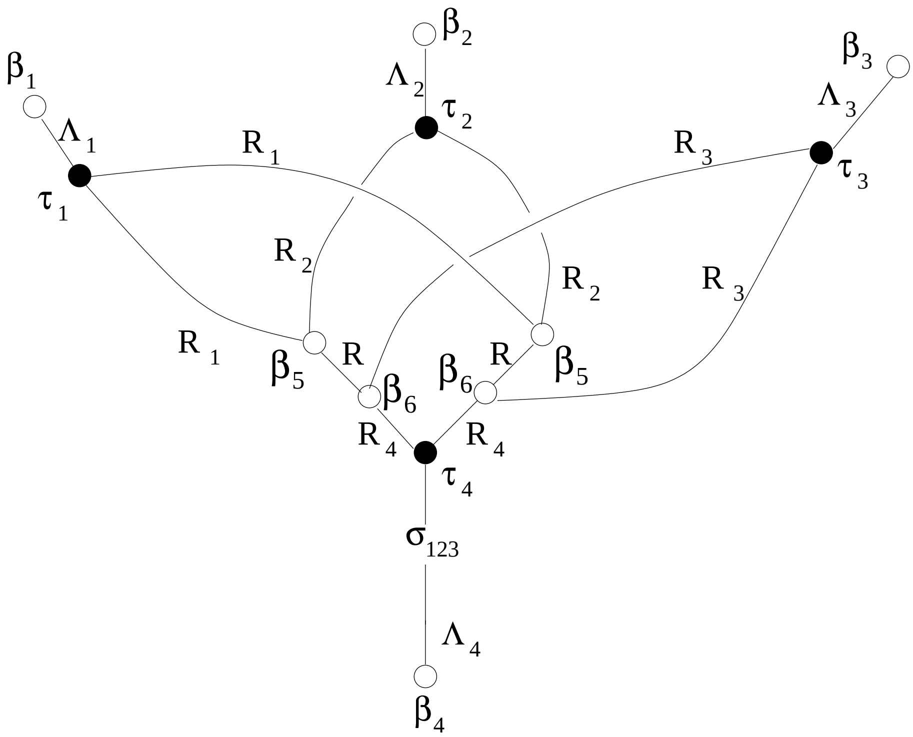

Figure 6: Splitting the 4-valent vertex into trivalents

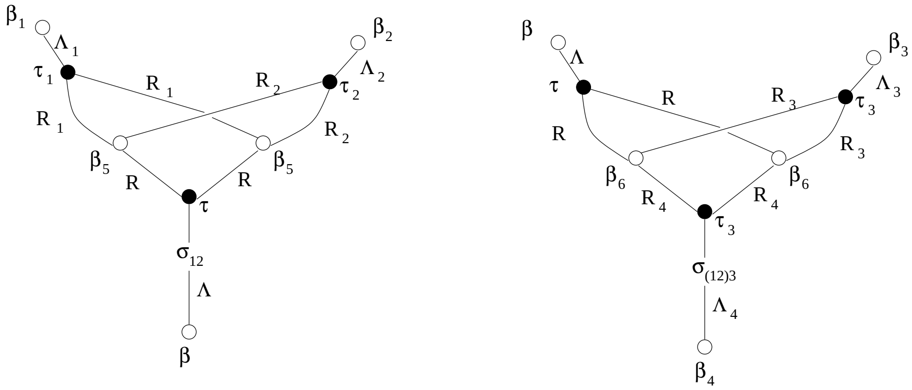

Figure 7: 4-point as product of 3-points

The relation to one-dimensional matrix models described in section 7.3 may also be helpful in uncovering the topological meaning.

6. Finite N projector on traces and dual bases

We have found that the complete set of holomorphic gauge invariant operators can be described in terms of a trace basis $\mathrm{tr}(\alpha\, \mathbf{X}^{\mu})$ or in terms of a basis $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ which diagonalises the two-point function. In this section we will use the index B for the set $(\Lambda, R, \beta, \tau)$. The content μ and the corresponding symmetry group H_μ are understood to be fixed. We will use the index A for equivalence classes of α under conjugation by permutations $\gamma \in H_\mu$. So we are using the notation $\mathcal{O}_A$ for the traces and $\mathcal{O}_B$ for the orthogonal basis. We have

$$\begin{aligned} \mathcal{O}_B &= \sum_A F_{BA} \mathcal{O}_A \\ \mathcal{O}_A &= \sum_B G_{AB} \mathcal{O}_B \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} F_{BA} &= \frac{D^R_{pq}(\alpha)}{n!} \; B_{j\beta} \; S^{\tau,\Lambda}{}_{j}{}^{R}{}_{p}{}^{R}{}_{q} \\ G_{AB} &= d_R D^R_{pq}(\alpha) \; B_{j\beta} \; S^{\tau,\Lambda}{}_{j}{}^{R}{}_{p}{}^{R}{}_{q} \end{aligned} \tag{6.2}$$

As a side remark, note that if we rescale $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ to have a $\sqrt{\frac{d_R}{n!}}$ instead of $\frac{1}{n!}$ then $F_{BA} = G_{AB}$. We can show, from (6.2) that

$$\sum_A F_{B_1 A} G_{A B_2} = \delta(B_1, B_2) \tag{6.3}$$

Indeed

$$\begin{aligned} &\sum_\alpha \frac{d_R}{n!} D^{R_1}_{p_1 q_1}(\alpha) B_{j_1\beta_1} S^{\tau_1,\Lambda_1}{}_{j_1}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1} D^{R_2}_{p_2 q_2}(\alpha) B_{j_2\beta_2} S^{\tau_2,\Lambda_2}{}_{j_2}{}^{R_2}{}_{p_2}{}^{R_2}{}_{q_2} \\ &\qquad = \delta^{R_1 R_2} S^{\tau_1,\Lambda_1}{}_{j_1}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1} S^{\tau_2,\Lambda_2}{}_{j_2}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1} B_{j_1\beta_1} B_{j_2\beta_2} \delta_{j_1 j_2} \\ &\qquad = \delta^{R_1 R_2} \delta_{\tau_1 \tau_2} \delta^{\Lambda_1 \Lambda_2} \delta_{\beta_1 \beta_2} \\ &\qquad = \delta(B_1, B_2) \end{aligned} \tag{6.4}$$

We used the orthogonality relations (A.3), (A.6), (2.21). Consider now

$$\begin{aligned} \sum_B G_{A_1 B} F_{B A_2} &= \sum_{R,\Lambda,\beta,\tau} \frac{d_R}{n!} D^{R_1}_{p_1 q_1}(\alpha) B_{j_1\beta_1} S^{\tau_1,\Lambda_1}{}_{j_1}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1} D^{R_2}_{p_2 q_2}(\alpha) B_{j_2\beta_2} S^{\tau_2,\Lambda_2}{}_{j_2}{}^{R_2}{}_{p_2}{}^{R_2}{}_{q_2} \\ &= \sum_{R,\Lambda,\tau} \frac{d_R}{n!} D^{R_1}_{p_1 q_1}(\alpha_1) D^{R_2}_{p_2 q_2}(\alpha_1) D^{\Lambda}_{j_1 j_2}(\Gamma) S^{\tau_1,\Lambda_1}{}_{j_1}{}^{R_1}{}_{p_1}{}^{R_1}{}_{q_1} S^{\tau_2,\Lambda_2}{}_{j_2}{}^{R_2}{}_{p_2}{}^{R_2}{}_{q_2} \\ &= \sum_R \sum_\gamma \frac{d_R}{|H|} D^R_{p_1 q_1}(\alpha_1) D^R_{p_2 q_2}(\alpha_2) D^R_{p_1 p_2}(\gamma) D^R_{q_1 q_2}(\gamma) \\ &= \frac{1}{|H|} \sum_\gamma \sum_R \frac{d_R}{n!} \chi_R(\alpha_1 \gamma \alpha_2 \gamma^{-1}) \end{aligned} \tag{6.5}$$

We first recognised the product of branching coefficients as the matrix element of a projector as in (2.20). Then we used the identity of the expressions in (3.35) and (3.37). When $n < N$, the sum over R runs over all irreps of S_n and we have

$$\begin{aligned} \sum_B G_{A_1 B} F_{B A_2} &= \frac{1}{|H|} \sum_\gamma \delta(\alpha_1 \gamma \alpha_2 \gamma^{-1}) \\ &= \delta(A_1, A_2) \qquad\qquad \text{for large N} \end{aligned} \tag{6.6}$$

When the large N condition is violated, i.e. when $n > N$, the sum over representations only runs over Young diagrams with first column $c_1(R)$ obeying $c_1(R) \leq N$, which does not exhaust the irreps of S_n. In this case we check that

$$P_{A_1 A_2} \equiv \frac{1}{|H|} \sum_{\gamma} \sum_{R: c_1(R) \leq N} \frac{d_R}{n!} \chi_R(\alpha_1 \gamma \alpha_2 \gamma^{-1}) \tag{6.7}$$

is a symmetric projector

$$\begin{aligned} P_{A_1 A_2} &= P_{A_2 A_1} \\ \sum_{A_3} P_{A_1 A_2} P_{A_2 A_3} &= P_{A_1 A_3} \end{aligned} \tag{6.8}$$

To summarise

$$\sum_{B} G_{A_1 B} F_{B A_2} = P_{A_1 A_2} \tag{6.9}$$

where P is the identity at large N, but is a projector at finite N. This projector has the property that it leaves the trace basis invariant

$$\sum_{A_2} P_{A_1 A_2} \mathcal{O}_{A_2} = \mathcal{O}_{A_1} \tag{6.10}$$

This can be obtained directly by using the Schur-Weyl duality relation (3.18) and recalling that for $\mathrm{U}(N)$ the Young diagrams are restricted to have first column of length less than or equal to N.

$$\begin{aligned} \mathrm{tr}(\alpha \mathbf{X}^{\mu}) &= \sum_{R: c_1(R) \leq N} \mathrm{tr}(P_R \alpha \mathbf{X}^{\mu}) \\ &= \sum_{R: c_1(R) \leq N} \sum_{\alpha_1} \frac{d_R}{n!} \chi_R(\alpha_1) \, \mathrm{tr}(\alpha_1 \alpha \mathbf{X}^{\mu}) \\ &= \sum_{R: c_1(R) \leq N} \frac{d_R}{n!} \chi_R(\alpha_1 \alpha^{-1}) \, \mathrm{tr}(\alpha_1 \mathbf{X}^{\mu}) \\ &= \frac{1}{|H_\mu|} \sum_{R: c_1(R) \leq N} \sum_{\alpha, \gamma} \frac{d_R}{n!} \chi_R(\gamma \alpha_1 \gamma^{-1} \alpha^{-1}) \, \mathrm{tr}(\alpha_1 \mathbf{X}^{\mu}) \end{aligned} \tag{6.11}$$

In the last line we used the invariance of $\mathrm{tr}(\alpha_1 \mathbf{X}^{\mu})$ under conjugation by γ. To go to the form in (6.10) simply recall that we are using A for the H_μ-equivalence classes of α. A similar discussion of finite N projectors appears in the context of 2D Yang Mills in [20].

It is instructive to define a *dual basis* to the traces [21, 22]

$$\mathcal{O}^*_A = \sum_{B} \frac{F_{BA}}{\langle B B^{\dagger} \rangle} \mathcal{O}_B = \sum_{B, A_1} \frac{F_{BA}}{\langle B B^{\dagger} \rangle} F_{B A_1} \; \mathcal{O}_{A_1} \tag{6.12}$$

It follows that

$$\begin{aligned} \langle \mathcal{O}^*_{A_1} \mathcal{O}^{\dagger}_{A_2} \rangle &= \sum_{B_1, B_2} \frac{F_{B_1 A_1} G_{A_2 B_2}}{\langle B_1 B_1^{\dagger} \rangle} \langle \mathcal{O}_{B_1} \mathcal{O}^{\dagger}_{B_2} \rangle \\ &= P_{A_1 A_2} \end{aligned} \tag{6.13}$$

7. Applications to gauge-gravity duality

7.1 Chiral ring; N particles in 3D SHO

Consider the space of holomorphic operators contructed from X, Y, Z. We have given a basis labelled by $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$. In this basis finite N effects are simply encoded by the the condition that the Young diagram R has no more than N rows. By the operator-state correspondence of CFT, these holomorphic gauge invariant operators map to a vector space V of states. Now set to zero the commutators.

$$[X,Y] = [Y,Z] = [X,Z] = 0 \tag{7.1}$$

A subspace of the space of operators will vanish. This gives a subspace of V. Let us call this subspace U. Given a subspace, we can define a quotient space V/U defined as the set of equivalence classes of vectors modulo addition of any vector in U. We have an exact sequence

$$0 \to U \to V \to V/U \to 0 \tag{7.2}$$

Note that we do not have a natural map from V/U to V. V/U is not a subspace of V. The quotient space corresponds by the operator-state relation to the chiral ring: operators modulo the equivalence of setting commutators to zero. There is no natural map from V/U to V because there is no fixed choice for the representatives of the equivalence class, so that for the chiral ring we could choose say the ordered trace, e.g. $\mathrm{tr}(XYYZ)$, or the symmetrised trace, e.g. $\mathrm{Str}(XYYZ)$.

By choosing some representatives of the equivalence classes, we can define a map from V/U to V. This map is not uniquely defined. But this won't matter. Having chosen the representatives we have an isomorphism between V and $U \oplus V/U$. There is a basis $\tilde{A} = (\tilde{A}_s, \tilde{A}_a)$ where s runs over the equivalence classes, and a over a basis for the null-space. The operators $\mathcal{O}_{A_s}$ are chosen representatives of the chiral ring. The operators $\mathcal{O}_{A_a}$ span the operators which vanish when commutators are set to zero.

Let us now restrict our considerations to $n < N$. Let $\mathcal{O}_A$ be any convenient basis, say the trace basis. It can be enumerated using Pólya theory. For small n it is easy to write the trace basis by hand. Since we know the transformation between the trace basis and the orthogonal basis $\mathcal{O}_B$ explicitly, we have duals of the trace basis according to (6.12).

The $\mathcal{O}_{\tilde{A}}$ are related to $\mathcal{O}_A$ by some invertible transformation

$$\mathcal{O}_{\tilde{A}} = \sum_A S_{\tilde{A}A} \mathcal{O}_A \tag{7.3}$$

The inverse $T_{A\tilde{A}}$ satisfies

$$\begin{aligned} \sum_A S_{\tilde{A}A} T_{A\tilde{A}'} &= \delta_{\tilde{A}\tilde{A}'} \\ \sum_{\tilde{A}} T_{A\tilde{A}} S_{\tilde{A}A'} &= \delta_{A'A} \end{aligned} \tag{7.4}$$

The dual basis for the $\mathcal{O}_{\tilde{A}}$ is given by

$$\mathcal{O}^*_{\tilde{A}} = \sum_A T_{A\tilde{A}} \mathcal{O}^*_A \tag{7.5}$$

After using the expression (6.12) for $\mathcal{O}^*_A$ we have

$$\mathcal{O}^*_{\tilde{A}} = \sum_{B,A,A_1} \frac{F_{BA} T_{A\tilde{A}} F_{BA_1}}{\langle BB^\dagger \rangle} \, \mathcal{O}_{A_1} \tag{7.6}$$

Indeed we check

$$\begin{aligned} \langle \mathcal{O}^*_{\tilde{A}} \mathcal{O}^\dagger_{\tilde{A}'} \rangle &= \sum_{A,A'} T_{A\tilde{A}} S_{\tilde{A}'A'} \langle \mathcal{O}^*_A \mathcal{O}^\dagger_{A'} \rangle \\ &= \sum_A S_{\tilde{A}'A} T_{A\tilde{A}} = \delta_{\tilde{A}'\tilde{A}} \end{aligned} \tag{7.7}$$

For $n > N$, the H_μ-equivalence classes of traces form an overcomplete set of operators. They have to be projected using the finite N projector (6.7) in section 6. Nevertheless the outcome of the above discussion (7.6) can be reproduced in the finite N case. As before, after choosing representatives of the chiral ring, we have a basis of operators $\mathcal{O}_{\tilde{A}}$, which includes those vanishing when commutators are set to zero, labelled by $\tilde{A}_a$, along with the representatives of chiral ring $\mathcal{O}_{\tilde{A}_s}$. There is an invertible transformation which relates this basis to the orthogonal basis

$$\mathcal{O}_{\tilde{A}} = \sum_B D_{\tilde{A}B} \mathcal{O}_B \tag{7.8}$$

$$\mathcal{O}_B = \sum_{\tilde{A}} D^{-1}_{B\tilde{A}} \mathcal{O}_{\tilde{A}} \tag{7.9}$$

The sum over B is constrained by $c_1(R) \leq N$. A dual basis for the $\mathcal{O}_{\tilde{A}}$ can be given as

$$\mathcal{O}^*_{\tilde{A}} = \sum_B D^{-1}_{B\tilde{A}} \frac{\mathcal{O}_B}{\langle BB^\dagger \rangle} \tag{7.10}$$

It is easy to check that

$$\langle \mathcal{O}^*_{\tilde{A}} \mathcal{O}^\dagger_{\tilde{A}'} \rangle = \delta_{\tilde{A}\tilde{A}'} \tag{7.11}$$

Now we would like to express the duals in (7.10) in terms of a transformation of the redundant, non-orthogonal, yet convenient trace basis $\mathcal{O}_A$. Expanding $\mathcal{O}_B$ in terms of $\mathcal{O}_A$ in (7.8) and using (7.9) in the equation for $\mathcal{O}_A$ in terms of $\mathcal{O}_B$, we get

$$\begin{aligned} \mathcal{O}_{\tilde{A}} &= \sum_A S_{\tilde{A}A} \mathcal{O}_A \\ \mathcal{O}_A &= \sum_{\tilde{A}} T_{A\tilde{A}} \mathcal{O}_{\tilde{A}} \end{aligned} \tag{7.12}$$

where S, T are defined by

$$S_{\tilde{A}A} = \sum_B D_{\tilde{A}B} F_{BA}$$
$$T_{A\tilde{A}} = \sum_B G_{AB} D^{-1}_{B\tilde{A}} \tag{7.13}$$

Unlike the large N case, S, T are not inverses of each other, but instead we can derive

$$\sum_A S_{\tilde{A}_1 A} T_{A\tilde{A}_2} = \delta_{\tilde{A}_1\tilde{A}_2}$$
$$\sum_{\tilde{A}} T_{A_1\tilde{A}} S_{\tilde{A}A_2} = P_{A_1A_2} \tag{7.14}$$

where $P_{A_1A_2}$ is the finite N projector defined in section 6. Using (7.13) we can obtain an expression for D^{-1} as

$$D^{-1}_{B\tilde{A}} = \sum_A F_{BA} T_{A\tilde{A}} \tag{7.15}$$

This allows us to rewrite (7.10) as

$$\mathcal{O}^*_{\tilde{A}} = \sum_{B,A_1,A} \frac{F_{BA_1} T_{A_1\tilde{A}} F_{BA}}{\langle BB^\dagger\rangle} \mathcal{O}_A \tag{7.16}$$

This is identical in form to (7.6) but now understood in a finite N context.

The duality equation (7.11) can be unpacked to give

$$\langle \mathcal{O}^*_{\tilde{A}_s} \mathcal{O}^\dagger_{\tilde{A}_a}\rangle = 0$$
$$\langle \mathcal{O}^*_{\tilde{A}_{s_i}} \mathcal{O}^\dagger_{\tilde{A}_{s_j}}\rangle = \delta_{ij}$$
$$\langle \mathcal{O}^*_{\tilde{A}_{a_i}} \mathcal{O}^\dagger_{\tilde{A}_{a_j}}\rangle = \delta_{ij} \tag{7.17}$$

From the first equation it is clear that the space of operators spanned by $\mathcal{O}^*_{\tilde{A}_{s_i}}$ gives the orthogonal complement of the null space U. This orthogonal complement is unique. It does not depend on the choice of representatives of the chiral ring. Note that we have characterised the orthogonal complement without having to diagonalise the metric on the space of states spanning U. We do not expect this orthogonality relation to be renormalised and indeed this has been shown at 1 loop for certain quarter BPS operators in [23]. The operators containing commutators (corresponding to the subspace U) are the descendants of long operators at all values of the coupling (they will mix amongst themselves but the space U remains invariant). Therefore since we do not expect mixing between short operators and long operators, the space orthogonal to U will also not change.

It is very interesting to consider the metric

$$\langle \mathcal{O}^*_{\tilde{A}_{s_i}} \mathcal{O}^{\dagger *}_{\tilde{A}_{s_j}}\rangle \equiv G^*_{ij} \tag{7.18}$$

on the short operators. We have

$$\begin{aligned}\langle \mathcal{O}^*_{\tilde{A}} \mathcal{O}^{\dagger *}_{\tilde{A}'} \rangle &= \sum_B \frac{D^{-1}_{\tilde{A}B} D^{-1}_{\tilde{A}'B}}{\langle BB^\dagger \rangle} \\ &= \sum_{B,A_1,A_2} \frac{F_{BA_1} F_{BA_2}}{\langle BB^\dagger \rangle} T_{A_1\tilde{A}} T_{A_2\tilde{A}'} \end{aligned} \tag{7.19}$$

When $\tilde{A}, \tilde{A}'$ are restricted to the $\tilde{A}_{s_i}, \tilde{A}_{s_j}$ i.e. to the subspace V/U, then (7.19) gives G^*. The metric (7.18) should not be renormalised. These non-renormalisation theorems have been shown at one loop by direct calculation in the case of quarter BPS operators of dimension up to eight [24, 23]. They can be proved using the insertion formula, first applied to the correlation functions of half-BPS in [25, 26] and applied to more general protected operators such as the quarter- and eighth-BPS operators considered here in [27, 28]. Furthermore it was shown there that the extremal higher-point correlators should also be non-renormalised. It would be very interesting however to have some checks of these non-renormalisation theorems by direct calculation at higher loops and for more general operators. The other equations in (7.17) will receive corrections at higher orders in g^2_{YM}. Further progress on the metric (7.18) can be obtained by finding simplifications of the quantity

$$\sum_B \frac{F_{BA_1} F_{BA_2}}{\langle BB^\dagger \rangle} \tag{7.20}$$

appearing in (7.19).

For large N, i.e. when the dimensions of the operators n is less than N, we can choose as representatives of the chiral ring, the products symmetrised traces, or we could choose products of ordered traces $\mathrm{tr}(X^* Y^* Z^*)$. Either choice will lead to the same (unique) orthogonal complement of the null-space by the above construction. For finite N, the essential part of the above story goes through, but it is worth spelling out where the additional subtleties lie. There is still a well-defined subspace U corresponding to vanishing operators when commutators are set to zero. There is a well-defined quotient space V/U. By choosing representatives of the chiral ring, we have an isomorphism from V to $V \oplus V/U$. In this case we are not committing ourselves to choosing symmetrised traces as representatives, since there could be complicated relations among them at finite N. All we need is that there exists some choice of representatives. The form of possible convenient choices will follow by looking more explicitly at the form of the finite N projector (6.7).

The quarter and eighth-BPS gauge invariant operators should be related to giant gravitons generalizing the analogous connection in the half-BPS case. It has been argued that the physics of the eighth-BPS giants [29] is given by the dynamics of N particles in a 3D simple harmonic oscillator [30 – 32]. States of harmonic oscillator system

$$\prod_{i=1}^N a^{i\,\dagger}_{n_{i1},n_{i2},n_{i3}} |0\rangle \tag{7.21}$$

The index i labels the particles. The natural numbers (n_{i1}, n_{i2}, n_{i3}) label the excitations along the x, y, z direction for the i'th particle. When we take an overlap of such a state with excitations n_{ia} with the conjugate of another state with excitations n'_{ia} we get an answer proportional to

$$\prod_{ia} \delta(n_{ia}, n'_{ia}) \tag{7.22}$$

In the leading large N (planar) limit there is a simple map between the harmonic oscillator states and gauge invariant operators, which preserves the metric. The above SHO states can be associated with

$$\prod_{i=1} \mathrm{Str}(X^{n_{i1}} Y^{n_{i2}} Z^{n_{i3}})|0\rangle \tag{7.23}$$

In the leading large N (planar) limit, it does not actually matter whether we choose symmetrised traces or ordered traces. This because different trace structures do not mix, and mixings between different orderings within a trace are also subleading in $1/N$. With either choice, we have the orthogonality (7.22) following from correlators of gauge invariant operators. But this does not work at subleading orders in $1/N$ or at finite N.

The discussion above, in terms of duals, gives a formula (7.19) for the metric G^* for the short representations. This formula works to all orders in the $1/N$ expansion or at finite N. The formulae given above in terms of $F_{BA}, G_{AB}, S_{\tilde{A}A}, T_{A\tilde{A}}$ give a way to work out G^*. After one diagonalises G^*, there should be a map from the orthogonal basis of short operators to the SHO states (7.21). The correct explicit form of such a map remains an interesting open problem. We have only offered a recipe based on the F, G, S, T which can be applied in a case by case basis to write down G^*.

We also expect that extremal correlators

$$\langle \mathcal{O}^*_{A_{s_1}}(x_1) \mathcal{O}^*_{A_{s_2}}(x_2) \cdots \mathcal{O}^*_{A_{s_n}}(x_n) \mathcal{O}^{\dagger *}_{A_{s_{n+1}}}(x_{n+1}) \rangle \tag{7.24}$$

are non-renormalised. They are expected to receive no corrections away from their zero-coupling values, either at 1-loop or higher loops or non-perturbatively. At strong coupling they are related to supersymmetric eighth-BPS giant gravitons. The correct map between linear combinations of operators $\mathcal{O}^*_{A_s}$ and observables in the N-particles in 3D-SHO should also allow a matching of these extremal correlators.

A better understanding of G^* should also help in making contact with quarter and eighth-BPS generalisations of LLM geometries. These have been discussed recently in [33, 34]. For example the geometric description of boundary conditions required for regularity of solutions, given in [33] should have a counterpart in the parametrisation of corresponding gauge invariant operators with metric (7.18). Combining the results on three-point functions in 5 along with the projection to the chiral ring in 6 should lead to the three-point functions relevant to the strong coupling limit. Following the arguments developed in [35] these three-point functions contain detailed information about the generalised LLM geometries.

7.2 Supergravity and $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$

In the above discussion we have discussed the spacetime interpretation of the zero-coupling gauge-invariant operators by first considering their weak-coupling counterparts, which involves a projection to the orthogonal subspace to the operators which vanish when commutators are set to zero. A direct space-time interpretation of the zero-coupling diagonal basis $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ would be desirable. Of course by AdS/CFT duality such a description must exist, but because $R = (g_sN)^{1/4}l_s$ this description will be at strong curvature, where SUGRA methods are not applicable. In this tensionless limit of string theory there is also an interesting supersymmetric version of the Higgs effect discussed in [36 − 39], which relates to some of the holomorphic operators we have considered becoming part of long representations.

We may consider the S-duality $g_s \to \frac{1}{g_s}$ which will map us to the large radius regime. In usual discussions of AdS/CFT, one uses the S-duality to fix $g_s < 1$. Any R is then accessible by tuning N. However, when the objects of interest include finite N effects as a function of coupling as well as a transition from zero to infinitesimal coupling, it is natural to consider the large g_s region at finite N. By the S-duality the jump in SUSY states from zero to infinitesimal coupling will then become a jump from finite radius to the flat space limit. It would be interesting to explore supergravity approaches to this regime, and perhaps in this context the zero-coupling diagonalisation, and the associated parameters $(\Lambda, \mu, R, \beta, \tau)$ can be interpreted in terms of parameters of spacetime solutions.

7.3 Reduced multi-matrix model

There is a reduced complex multi-matrix model obtained by reducing the free action for the scalars of 4D $N = 4$ SYM on $S^3 \times R$.

$$\int dt \, \sum_a \mathrm{tr} \, \, \partial_t X_a \partial_t X_a^\dagger + \mathrm{tr} \, X_a X_a^\dagger \tag{7.25}$$

The Gauss Law constraint requires a projection to gauge invariant (traced) states. Following [1] (see also [40 − 42]), the Hamiltonian is

$$H = \sum_a \mathrm{tr}(A_a^\dagger A_a + B_a^\dagger B_a) \tag{7.26}$$

The index a is a flavour index which runs from 1 to 3. The operators A_a, B_a also carry matrix indices and obey

$$\begin{aligned} [A_{a}{}^i_j, A^{\dagger k}_{b\,l}] &= \delta_{ab}\delta^i_l\delta^k_j \\ [B_{a}{}^i_j, B^{\dagger k}_{b\,l}] &= \delta_{ab}\delta^i_l\delta^k_j \\ [A, A] = [B, B] &= 0 \end{aligned} \tag{7.27}$$

The BPS states satisfy $E = J_1 + J_2 + J_3$. The $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ charges are

$$J_a = \mathrm{tr}(A_a^\dagger A_a - B_a^\dagger B_a) \tag{7.28}$$

The BPS states are constructed by acting with $A^\dagger$ only, and no $B^\dagger$. We can construct operators in this matrix quantum mechanics by replacing the $\mathbf{X}^\mu$ of earlier sections with sequences built from $A^\dagger$. The μ_1 copies of X_1, μ_2 of X_2 etc are replaced by μ_1 of $A_1^\dagger$ etc. Our results on the diagonality of correlators imply immediately that

$$\langle 0|\mathcal{O}^{\Lambda_1\mu^{(1)},R_1}_{\beta_1,\tau_1}(A)\mathcal{O}^{\Lambda_2\mu^{(2)},R_2}_{\beta_2,\tau_2}(A^\dagger)|0\rangle = \delta^{\mu^{(1)}\mu^{(2)}}\delta^{\Lambda_1\Lambda_2}\delta^{R_1R_2}\delta_{\beta_1\beta_2}\delta_{\tau_1\tau_2} \tag{7.29}$$

Similarly our results on extremal correlators have counterparts in this multi-matrix quantum mechanics.

From the matrix oscillators we can construct composites E_{ab}, T^i_j which obey $\mathrm{U}(N)$ and $\mathrm{U}(M)$ Lie algebra relations under commutation.

$$\begin{aligned} E_{ab} &= A^{\dagger\,i}_{a\,j}A_{b\,i}^{\;j} + B^{\dagger\,i}_{a\,j}B_{b\,i}^{\;j} \\ T^i_j &= A^{\dagger\,i}_{a\,k}A_{a\,j}^{\;k} + B^{\dagger\,i}_{a\,k}B_{a\,j}^{\;k} \end{aligned} \tag{7.30}$$

Higher Hamiltonians can be constructed as $\mathrm{tr}(T^n)$ and $\mathrm{tr}(E^m)$ which commute with the Hamiltonian H. Their eigenvalues, when acting on states, $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}(A^\dagger)|0\rangle$ are invariants (Casimirs) depending on R, Λ respectively. It is an interesting question whether there are conserved charges which measure τ and β directly. Such conserved charges may help in finding a spacetime interpretation following the discussion of [43] in the half-BPS case. These multi-matrix models have also been discussed in [44] where they were related to Calogero models. Expressing our diagonal operators in the fermionic variables may be useful in clarifying the integrable structure.

8. Conclusions and open questions

We have given a diagonal basis for correlators in an $N \times N$ matrix theory with $\mathrm{U}(M)$ global symmetry. This has applications in the zero coupling limit of $\mathcal{N} = 4$ SYM. Holomorphic operators in the three complex spacetime-scalar matrix fields are BPS at zero coupling. For the two-point functions of these holomorphic operators with their conjugates, we have a diagonal basis. Some of these operators become part of long representations at weak coupling. We have given a characterisation of the orthogonal space to these long operators, which are genuine eighth-BPS operators. Our discussion is valid at finite N. This gives a framework which can be used for the comparison of computations in the N-particle 3D-SHO, which comes from eigenvalues dynamics, with the gauge invariant operators. In particular extremal correlators of the genuine eighth-BPS operators computed from the gauge-invariant set-up of this paper should agree with those computed from eigenvalue dynamics. Explicit formulae for the two-point functions of these genuine-BPS operators can be obtained from this paper, but a natural diagonalisation on this subspace remains an open problem. This will be useful in setting up a comparison with results from eigenvalue dynamics. It should also allow a better understanding of the mapping from the geometry of giant graviton moduli spaces to the space of diagonal gauge-invariant eighth-BPS operators at finite N, generalising the geometrical explanation [3] of the stringy exclusion principle [45].

It would be desirable to have a spacetime interpretation for the diagonal basis in the full set of holomorphic operators. By AdS/CFT there should certainly be a stringy interpretation. Whether there is an interpretation in the supergravity limit is a very interesting question. We have speculated, using S-duality that the answer is positive. Progress on these issues would be fascinating because it would give a spacetime interpretation to the parameter τ which runs over the Clebsch-multiplicities of symmetric groups. A simple algorithmic method for determining these multiplicities in terms of manipulating boxes of Young diagrams, analogous to the one available for Littlewood-Richardson coefficients, is still an open problem in symmetric groups. A spacetime interpretation of $\mathcal{O}^{\Lambda,\mu,R}_{\beta,\tau}$ might provide stringy insights into this problem.

Many of the techniques in this paper should extend to other gauge groups. The SU(N) gauge group case could be developed along the lines of [13, 46, 21] and orthogonal and symplectic groups along the lines of [47]. Another generalisation is to consider operators made from X, Y, Z along with $X^\dagger, Y^\dagger, Z^\dagger$, and to organise them according to the degree of singularity in their short distance subtractions. For the case of $X, X^\dagger$ this was done in [48] using Brauer algebras. We can also consider the more general protected operators defined in [49]. Another interesting line of research is to clarify the dynamics of strings connected to eighth-BPS giant gravitons in gauge theory and their connection to space-time geometry. A lot of progress in this area has been made for the half-BPS case (see for example for some of the initial developments and further references [50 – 54]).

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A. Formulae

The matrices of any representation satisfy the following property, which follows from Schur's Lemma

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma) D^S_{lk}(\sigma^{-1}) = \frac{n!}{d_R}\delta^{RS}\delta_{ik}\delta_{jl} \tag{A.1}$$

We will use orthogonal representation matrices, where orthogonality guarantees that

$$D^R_{ij}(\sigma^{-1}) = D^R_{ji}(\sigma) \tag{A.2}$$

This means that for orthogonal matrices equation (A.1) becomes

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d_R}\delta^{RS}\delta_{ik}\delta_{jl} \tag{A.3}$$

A.1 Clebsch-Gordan coefficients for the tensor product

The Clebsch-Gordan coefficients allow us to write operators in the (inner) tensor product space $S \otimes T$, such as $D^S_{ij}(\sigma)D^T_{kl}(\sigma)$, in terms of operators in a single representation R. We obtain them by inserting a complete sets of states

$$\begin{aligned} D^S_{ij}(\sigma)D^T_{kl}(\sigma) &= \langle S,i;T,k|\,\sigma\,|S,j;T,l\rangle \\ &= \sum_{R,R',\tau_R,\tau_{R'}} \langle S,i;T,k|R,\tau_R,a\rangle\,\langle R,\tau_R,a|\,\sigma\,\big|R',\tau_{R'},b\big\rangle\,\langle R',\tau_{R'},b|S,j;T,l\rangle \\ &= \sum_{R,\tau_R}\langle S,i;T,k|R,\tau_R,a\rangle\; D^R_{ab}(\sigma)\;\langle R,\tau_R,b|S,j;T,l\rangle \end{aligned} \tag{A.4}$$

where the label τ_R runs over the multiplicity of the appearance of R in the inner product $S\otimes T$. Since the representing matrices are real, the Clebsch-Gordan coefficients are also real, so we can write

$$S^{\tau_R,R}{}_{a}{}^{S}{}_{i}{}^{T}{}_{k} \equiv \langle S,i;T,k|R,\tau_R,a\rangle = \langle R,\tau_R,a|S,i;T,k\rangle \tag{A.5}$$

The Clebsch-Gordan coefficients satisfy the following orthogonality relations [8]

$$\sum_{j,k} S^{\tau_R,R}{}_{a}{}^{U}{}_{j}{}^{V}{}_{k}\; S^{\tau_S,S}{}_{b}{}^{U}{}_{j}{}^{V}{}_{k} = \delta^{RS}\delta^{\tau_R\tau_S}\delta_{ab} \tag{A.6}$$

$$\sum_{\tau_R}\sum_{R}\sum_{a} S^{\tau_R,R}{}_{a}{}^{U}{}_{i}{}^{V}{}_{j}\; S^{\tau_R,R}{}_{a}{}^{U}{}_{k}{}^{V}{}_{l} = \delta_{ik}\delta_{jl} \tag{A.7}$$

From (A.4) we can then derive

$$\sum_{j,l} D^S_{ij}(\sigma)D^T_{kl}(\sigma)\; S^{\tau_R,R}{}_{s}{}^{S}{}_{j}{}^{T}{}_{l} = \sum_{t} D^R_{ts}(\sigma) S^{\tau_R,R}{}_{t}{}^{S}{}_{i}{}^{T}{}_{k} \tag{A.8}$$

$$\sum_{\sigma} D^R_{ts}(\sigma)D^S_{ij}(\sigma)D^T_{kl}(\sigma) = \frac{n!}{d_R}\sum_{\tau_R} S^{\tau_R,R}{}_{t}{}^{S}{}_{i}{}^{T}{}_{k}\; S^{\tau_R,R}{}_{s}{}^{S}{}_{j}{}^{T}{}_{l} \tag{A.9}$$

Note that, by taking traces in (A.9) and using (A.6) we can recover $C(R,S,T)$ (C.3) which comes from the sum over τ_R.

B. Some basic linear algebra

We often meet the following simple situation in this paper. We have a vector space of dimension d, V^d, sitting inside a vector space of dimension D, V^D. We have a projector P_{ij} which projects onto the smaller vector space, ie

$$V^d = v_i : v_i = P_{ij}v_j\ . \tag{B.1}$$

Then if we can write the projector as

$$P_{ij} = \sum_{\beta=1}^{d} b_{i\beta}b_{j\beta} \tag{B.2}$$

where

$$\sum_i b_{i\beta}b_{i\beta'} = \delta_{\beta\beta'} \tag{B.3}$$

then the vectors $v_\beta := \sum_i b_{i\beta}v_i$ provide an orthonormal basis for the subspace V^d.

C. Counting proofs

C.1 Explicit gauge-invariant proof

The formula for the number of gauge-invariants operators (including multi-traces) made from fields μ_1 of X_1, μ_2 of X_2, $\ldots \mu_M$ of X_M is given by Pólya theory to be the coefficient of $x_1^{\mu_1} \cdots x_M^{\mu_M}$ in

$$\begin{aligned}
&\prod_{k=1}^{\infty} \frac{1}{1-(x_1^k + \cdots + x_M^k)} \qquad\qquad (C.1)\\
&\quad = \sum_{i_1, i_2, \ldots} (x_1 + \cdots + x_M)^{i_1} \, (x_1^2 + \cdots + x_M^2)^{i_2} \cdots \\
&\quad = \sum_{i_1, i_2, \ldots} \left(\sum_{i_1^1, \ldots i_1^M \mid \sum_\alpha i_1^\alpha = i_1} \frac{i_1!}{i_1^1! \cdots i_1^M!} x_1^{i_1^1} \ldots x_M^{i_1^M} \right) \times \\
&\qquad\qquad \times \left(\sum_{i_2^1, \ldots i_2^M \mid \sum_\alpha i_2^\alpha = i_2} \frac{i_2!}{i_2^1! \cdots i_2^M!} x_1^{2 i_2^1} \ldots x_M^{2 i_2^M} \right) \cdots \\
&\quad = \sum_{i_1, i_2, \ldots} \sum_{\{i_1^\alpha\}, \{i_2^\alpha\}, \ldots \mid \sum_\alpha i_1^\alpha = i_1, \sum_\alpha i_2^\alpha = i_2, \ldots} \frac{i_1!}{i_1^1! \cdots i_1^M!} \frac{i_2!}{i_2^1! \cdots i_2^M!} \cdots x_1^{i_1^1 + 2 i_2^1 + \cdots} \cdots x_M^{i_1^M + 2 i_2^M + \cdots}
\end{aligned}$$

In the third line we have used a generalised binomial expansion; in the fourth line we have collected coefficients of x_1, etc. So we pick out the coefficient with $\mu_1 = i_1^1 + 2i_2^1 + \cdots, \ldots \mu_M = i_1^M + 2i_2^M + \cdots$. This makes the coefficient

$$N(\mu_1, \ldots \mu_M) = \sum_{\{i_1^\alpha\}, \{i_2^\alpha\}, \ldots \mid \mu_k = i_1^k + 2 i_2^k + \cdots} \frac{(\sum_\alpha i_1^\alpha)!}{i_1^1! \cdots i_1^M!} \frac{(\sum_\alpha i_2^\alpha)!}{i_2^1! \cdots i_2^M!} \cdots \tag{C.2}$$

The coefficient $C(R, S, T)$ for T in the inner product for S_n representations $R \otimes S = \sum_T C(R, S, T) T$ is given by

$$C(R, S, T) = \sum_{\mathbf{i}} \frac{1}{|\mathrm{Sym}(C_{\mathbf{i}})|} \chi_R(C_{\mathbf{i}}) \chi_S(C_{\mathbf{i}}) \chi_T(C_{\mathbf{i}}) \tag{C.3}$$

Notice that it is symmetric in R, S, T. $C_{\mathbf{i}}$ represents the conjugacy class of S_n with i_1 1-cycles, i_2 2-cycles, $\ldots i_n$ n-cycles (of course we have $\sum_\alpha \alpha i_\alpha = n$, which is just the condition that $C_{\mathbf{i}}$ is in S_n). $|\mathrm{Sym}(C_{\mathbf{i}})|$ is the size of the symmetry group of the conjugacy class $C_{\mathbf{i}}$ and satisfies

$$|\mathrm{Sym}(C_{\mathbf{i}})| = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdots i_n! n^{i_n} = \frac{n!}{|[C_{\mathbf{i}}]|} \tag{C.4}$$

Here $|[C_{\mathbf{i}}]|$ is the size of the conjugacy class of $C_{\mathbf{i}}$.

We can immediately simplify $\sum_R C(R, R, \Lambda)$ using the orthogonality relation

$$\sum_R \chi_R(\rho) \chi_R(\tau) = |\mathrm{Sym}(\tau)| \;\; \delta([\tau] = [\rho]) \tag{C.5}$$

We get

$$\sum_R C(R,R,\Lambda) = \sum_{C_{\mathbf{i}}\in S_n} \chi_\Lambda(C_{\mathbf{i}}) \tag{C.6}$$

Now apply these formulae to the expression for $N(\mu_1,\dots \mu_M)$ in terms of representation-theoretic data

$$\begin{aligned}
&N(\mu_1,\dots \mu_M)\\
&\quad = \sum_R \sum_\Lambda C(R,R,\Lambda) g([\mu_1],\dots [\mu_M];\Lambda)\\
&\quad = \sum_\Lambda \sum_{C_{\mathbf{i}}\in S_n} \chi_\Lambda(C_{\mathbf{i}}) \frac{1}{\mu_1!\cdots \mu_M!} \sum_{\rho_1\in S_{\mu_1}} \cdots \sum_{\rho_M\in S_{\mu_M}} \chi_{[\mu_1]}(\rho_1)\cdots\chi_{[\mu_M]}(\rho_M)\chi_\Lambda(\rho_1\circ\cdots\circ\rho_M)
\end{aligned}$$

Here we have used the formula for Littlewood-Richardson coefficients

$$g(R_1,\dots R_k;T) = \frac{1}{n_1!\cdots n_k!} \sum_{\rho_1\in S_{n_1}} \cdots \sum_{\rho_k\in S_{n_k}} \chi_{R_1}(\rho_1)\cdots\chi_{R_k}(\rho_k)\chi_T(\rho_1\circ\cdots\circ\rho_k) \tag{C.7}$$

But of course we know that $\chi_{[\mu]}(\rho) = 1\ \forall\rho$ and we can perform the R sum

$$\begin{aligned}
N(\mu_1,\dots \mu_M) =& \sum_\Lambda \sum_{\mathbf{i}} \chi_\Lambda(C_{\mathbf{i}}) \frac{1}{\mu_1!\cdots \mu_M!} \sum_{\rho_1\in S_{\mu_1}} \cdots \sum_{\rho_M\in S_{\mu_M}} \chi_\Lambda(\rho_1\circ\cdots\circ\rho_M)\\
=& \frac{1}{\mu_1!\cdots \mu_M!} \sum_{\rho_1\in S_{\mu_1}} \cdots \sum_{\rho_M\in S_{\mu_M}} |\mathrm{Sym}(\rho_1\circ\cdots\circ\rho_M)|
\end{aligned} \tag{C.8}$$

If we sum over conjugacy classes $C_{\mathbf{i}^\alpha}$ in S_{μ_α} instead we get

$$\begin{aligned}
N(\mu_1,\dots \mu_M) &= \frac{1}{\mu_1!\cdots \mu_M!} \sum_{C_{\mathbf{i}^1}\in S_{\mu_1}} \cdots \sum_{C_{\mathbf{i}^M}\in S_{\mu_M}} |[C_{\mathbf{i}^1}]|\cdots|[C_{\mathbf{i}^M}]||\mathrm{Sym}(C_{\mathbf{i}^1}\circ\cdots\circ C_{\mathbf{i}^M})|\\
&= \sum_{C_{\mathbf{i}^1}\in S_{\mu_1}} \cdots \sum_{C_{\mathbf{i}^M}\in S_{\mu_M}} \frac{|\mathrm{Sym}(C_{\mathbf{i}^1}\circ\cdots\circ C_{\mathbf{i}^M})|}{|\mathrm{Sym}(C_{\mathbf{i}^1})|\cdots|\mathrm{Sym}(C_{\mathbf{i}^M})|}\\
&= \sum_{C_{\mathbf{i}^1}\in S_{\mu_1}} \cdots \sum_{C_{\mathbf{i}^M}\in S_{\mu_M}} \frac{(\sum_\alpha i_1^\alpha)!\cdots(\sum_\alpha i_n^\alpha)!}{i_1^1!\, i_2^1!\cdots i_n^1!\cdots i_1^M!\, i_2^M!\cdots i_n^M!}
\end{aligned} \tag{C.9}$$

Finally we identify this with the Pólya theory result (C.2). Note that the quantity summed here can also be recognised as the character of the representation of S_n induced from the trivial representation of the subgroup $S_{\mu_1}\times\cdots S_{\mu_M}$

$$\psi_\Lambda(C_{\mathbf{i}}) = \sum_\Lambda g([\mu_1],\dots[\mu_M];\Lambda)\chi_\Lambda(C_{\mathbf{i}}) \tag{C.10}$$

C.2 Example of the counting

Consider the counting for operators with fields XXY.

The relevant S_3 outer product is

$$[2]\circ[1] = [3]\oplus[2,1] \tag{C.11}$$

which gives us

$$\begin{aligned} N(2,1) &= \sum_R \sum_\Lambda C(R,R;\Lambda) g([2],[1];\Lambda) \\ &= \sum_R \{C(R,R;[3]) + C(R,R;[2,1])\} \end{aligned} \tag{C.12}$$

The relevant inner products are

$$\begin{aligned} [3]\otimes[3] &= [3] \\ [2,1]\otimes[2,1] &= [3]\oplus[2,1]\oplus[1,1,1] \\ [1^3]\otimes[1^3] &= [3] \end{aligned} \tag{C.13}$$

So we get

$$N(2,1) = 3+1 = 4 \tag{C.14}$$

This counts correctly for $\mathrm{tr}(XXY)$, $\mathrm{tr}(XX)\,\mathrm{tr}(Y)$, $\mathrm{tr}(XY)\,\mathrm{tr}(X)$ and $\mathrm{tr}(X)\,\mathrm{tr}(X)\,\mathrm{tr}(Y)$.

C.3 Trace counting

The counting formula is give by (C.9)

$$\begin{aligned} N(\mu_1,\dots\mu_M) &= \frac{1}{\mu_1!\cdots\mu_M!}\sum_{C_{\mathbf{i}^1}\in S_{\mu_1}}\cdots\sum_{C_{\mathbf{i}^M}\in S_{\mu_M}} |[C_{\mathbf{i}^1}]|\cdots|[C_{\mathbf{i}^M}]||\mathrm{Sym}(C_{\mathbf{i}^1}\circ\cdots\circ C_{\mathbf{i}^M})| \\ &= \frac{1}{|H|}\sum_{h\in H}|\mathrm{Sym}(h)| \end{aligned} \tag{C.15}$$

We need to count $\mathrm{tr}(\alpha X_1^{\mu_1}\cdots X_M^{\mu_M})$ up to the equivalence relation

$$\alpha \sim h^{-1}\alpha h, \quad h\in H \tag{C.16}$$

One way to find the number of equivalence classes is to sum over all α, dividing by the size of the equivalence class of each α

$$N(\mu_1,\dots\mu_M) = \sum_{[\alpha]} 1 = \sum_{\alpha\in S_n}\frac{1}{|[\alpha]|} = \sum_{\alpha\in S_n}\frac{1}{\text{no. of distinct } \beta\in S_n|\beta = h\alpha h^{-1} \text{ for } h\in H} \tag{C.17}$$

Naïvely we might think that

$$\big[\text{no. of distinct } \beta\in S_n|\beta = h\alpha h^{-1} \text{ for } h\in H\big] = |H| \tag{C.18}$$

We must however be careful: elements in H might also be in the symmetry group of α ($\mathrm{Sym}(\alpha) = \{\sigma\in S_n|\alpha = \sigma\alpha\sigma^{-1}\}$). To hit unique β's we must use not H but the coset $H/[\mathrm{Sym}(\alpha)\cap H]$

$$\big[\text{no. of distinct } \beta\in S_n|\beta = h\alpha h^{-1} \text{ for } h\in H\big] = |H/[\mathrm{Sym}(\alpha)\cap H]| = \frac{|H|}{|\mathrm{Sym}(\alpha)\cap H|} \tag{C.19}$$

Thus we get

$$
\begin{aligned}
N(\mu_1,\dots\mu_M) &= \frac{1}{|H|}\sum_{\alpha\in S_n}|\mathrm{Sym}(\alpha)\cap H| \\
&= \frac{1}{|H|}\sum_{\alpha\in S_n}\sum_{h\in H}[\text{s.t. }\ \alpha = h\alpha h^{-1}] \\
&= \frac{1}{|H|}\sum_{h\in H}\sum_{\alpha\in S_n}[\text{s.t. }\ h = \alpha h\alpha^{-1}] \\
&= \frac{1}{|H|}\sum_{h\in H}|\mathrm{Sym}(h)| \qquad \text{(C.20)}
\end{aligned}
$$

D. Calculating branching coefficients

$D^{\Lambda}_{j_1j_2}(\Gamma)$ projects onto a subspace of the S_n representation Λ with dimension $g(\mu;\Lambda)$; this subspace is given by the rows/columns of the matrix $D^{\Lambda}_{j_1j_2}(\Gamma)$. We want to find the branching coefficients $B_{j\beta}$ given by

$$D^{\Lambda}_{j_1j_2}(\Gamma) = \sum_{\beta} B_{j_1\beta}B_{j_2\beta} \qquad \text{(D.1)}$$

We work out some examples below.

D.1 Highest weight case

For the highest weight state with $\mu = \Lambda$ (for which $g(\mu;\Lambda) = 1$) Hamermesh's basis works such that

$$D^{\Lambda}_{j_1j_2}(\Gamma) = \delta_{j_11}\delta_{j_21} \qquad \text{(D.2)}$$

Thus the subspace is spanned by a single vector $B_j = \delta_{j1}$, which satisfies all the appropriate properties.

D.2 All fields different case

For $\mu_1 = 1,\dots\mu_M = 1$, i.e. all the fields are different, then $H = \mathrm{id}$ and $g(\mu;\Lambda) = d_\Lambda$

$$D^{\Lambda}_{j_1j_2}(\Gamma) = D^{\Lambda}_{j_1j_2}(\mathrm{id}) = \delta_{j_1j_2} \qquad \text{(D.3)}$$

The most obvious basis satisfying the correct properties is $B_{j\beta} = \delta_{j\beta}$ (see XYZ example below).

D.3 $\Lambda = [2,1]$

$$
\begin{aligned}
&\frac{1}{|H|}D^{\Lambda=[2,1],\mu=XXY}(\Gamma) = \frac{1}{2}D^{[2,1]}((1)(2)(3)+(12)(3)) = \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix} \\
&\frac{1}{|H|}D^{\Lambda=[2,1],\mu=XYY}(\Gamma) = \begin{pmatrix}\frac{1}{4}&\frac{\sqrt{3}}{4}\\ \frac{\sqrt{3}}{4}&\frac{3}{4}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix} \qquad \text{(D.4)}
\end{aligned}
$$

Note that for this last one the columns/rows of the matrix aren't independent (which concurs with the fact that $g = 1$), so the subspace is spanned by the first column say.

$$\frac{1}{|H|} D^{\Lambda=[2,1],\mu=XYZ}(\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{D.5}$$

D.4 $\Lambda = [3, 1]$

$$\frac{1}{|H|} D^{\Lambda=[3,1],\mu=XXXY}(\Gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{|H|} D^{\Lambda=[3,1],\mu=XXYY}(\Gamma) = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix} \tag{D.6}$$

$$\frac{1}{|H|} D^{\Lambda=[3,1],\mu=XYYY}(\Gamma) = \begin{pmatrix} \frac{1}{9} & \frac{\sqrt{2}}{9} & \frac{\sqrt{6}}{9} \\ \frac{\sqrt{2}}{9} & \frac{2}{9} & \frac{2\sqrt{3}}{9} \\ \frac{\sqrt{6}}{9} & \frac{2\sqrt{3}}{9} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \tag{D.7}$$

$$\frac{1}{|H|} D^{\Lambda=[3,1],\mu=XYYZ}(\Gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\frac{1}{|H|} D^{\Lambda=[3,1],\mu=XYZZ}(\Gamma) = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{pmatrix}$$

E. Examples of the operators

We will work out the operators $\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}$ given by (3.28)

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{n!} \sum_{\alpha} B_{j\beta}\, S^{\tau,\Lambda\ R\ R}_{\ \ j\ \ \ p\ q}\ D^R_{pq}(\alpha)\, \mathrm{tr}(\alpha\, \mathbf{X}^\mu) \tag{E.1}$$

E.1 XY

We have

$$D^{[2]}_{11}(\sigma) = 1 \qquad D^{[1,1]}_{11}(\sigma) = (-1)^\sigma \qquad \forall \sigma \tag{E.2}$$

For XY, $\mu = [1, 1]$, $H = S_1 \times S_1$ and $g([1], [1]; [2]) = g([1], [1]; [1, 1]) = 1$. This means there is only one possible value of β for each choice of Λ. $B_{11} = 1$ for all Λ.

Since there are no R for which $C(R, R, [1, 1])$ is non-zero, we only have $\Lambda = [2]$. For each R, $C(R, R, [2]) = 1$, so there is no τ multiplicity. Thus our two operators are

$$\mathcal{O}^{\Lambda=[2],R=[2]} = \frac{1}{2} \sum_\alpha D^{[2]}_{11}(\alpha)\, \mathrm{tr}(\alpha\ XY\ ) = \frac{1}{2}[\mathrm{tr}(X)\, \mathrm{tr}(Y) + \mathrm{tr}(XY)]$$

$$\mathcal{O}^{\Lambda=[2],R=[1,1]} = \frac{1}{2} \sum_\alpha D^{[1,1]}_{11}(\alpha)\, \mathrm{tr}(\alpha\ XY\ ) = \frac{1}{2}[\mathrm{tr}(X)\, \mathrm{tr}(Y) - \mathrm{tr}(XY)]$$

E.2 XXY

We have

$$D^{[3]}_{11}(\sigma) = 1 \qquad D^{[1,1,1]}_{11}(\sigma) = (-1)^{\sigma} \qquad \forall\sigma \tag{E.3}$$

and

$$D^{[2,1]}((1)(2)(3)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D^{[2,1]}((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$D^{[2,1]}((23)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad D^{[2,1]}((13)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$
$$D^{[2,1]}((123)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D^{[2,1]}((132)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

By the counting there are four possibilities: $\Lambda = [3]$ with $R = [3], [2,1], [1,1,1]$ and $\Lambda = [2,1]$ with $R = [2,1]$. For these possibilities $C(R, R, \Lambda) = 1$ so there is no τ multiplicity. $g([2],[1];[3]) = g([2],[1];[2,1]) = 1$ so there is no β multiplicity.

For XXY, $\mu = [2,1]$ and $H = S_2 \times S_1$. For $\Lambda = [3]$ $B_{11} = 1$ and for $\Lambda = [2,1]$ $B_{j1} = \delta_{j1}$, since this field content is a highest weight state for this Λ.

We can work out the relevant Clebsch-Gordan coefficients up to a sign using the identity (A.9) when τ only takes a single value

$$\left(S^{\Lambda}_{i}{}^{R}_{k}{}^{R}_{l}\right)^2 = \frac{d_\Lambda}{n!} \sum_{\sigma} D^{\Lambda}_{ii}(\sigma) D^{R}_{kk}(\sigma) D^{R}_{ll}(\sigma) \tag{E.4}$$

To fix the sign we must work out identities such as (A.8) explicitly. We get non-zero values

$$\begin{aligned} S^{[3]}_{1}{}^{R}_{k}{}^{R}_{l} &= \frac{1}{\sqrt{d_R}} \delta_{kl} \\ S^{[2,1]}_{1}{}^{[2,1]}_{1}{}^{[2,1]}_{1} &= \frac{1}{\sqrt{2}} \\ S^{[2,1]}_{1}{}^{[2,1]}_{2}{}^{[2,1]}_{2} &= -\frac{1}{\sqrt{2}} \end{aligned}$$

So we have

$$\begin{aligned} \mathcal{O}^{\Lambda=[3],R}_{\mu=[2,1]} &= \frac{1}{3!} \sum_{\alpha} S^{[3]}_{1}{}^{R}_{k}{}^{R}_{l} D^{R}_{kl}(\alpha) \operatorname{tr}(\alpha\, XXY\,) \\ &= \frac{1}{3!\sqrt{d_R}} \sum_{\alpha} \chi_R(\alpha) \operatorname{tr}(\alpha\, XXY\,) \\ \mathcal{O}^{\Lambda=[2,1],R=[2,1]}_{\mu=[2,1]} &= \frac{1}{3!} \sum_{\alpha} S^{[2,1]}_{1}{}^{[2,1]}_{k}{}^{[2,1]}_{l} D^{[2,1]}_{kl}(\alpha) \operatorname{tr}(\alpha\, XXY\,) \\ &= \frac{1}{3!} \sum_{\alpha} \left(\frac{1}{\sqrt{2}} D^{[2,1]}_{11}(\alpha) - \frac{1}{\sqrt{2}} D^{[2,1]}_{22}(\alpha) \right) \operatorname{tr}(\alpha\, XXY\,) \\ &= \frac{1}{3\sqrt{2}} \left[\operatorname{tr}(Y) \operatorname{tr}(XX) - \operatorname{tr}(X) \operatorname{tr}(XY) \right] \end{aligned}$$

E.3 XYY for $\Lambda = [2,1]$

$$\begin{aligned}\mathcal{O}^{\Lambda=[2,1],R=[2,1]}_{\mu=[1,2]} &= \sum_j \langle [2,1](S_3) \to S_1 \times S_2, \mathbf{1} | [2,1], j \rangle \mathcal{O}^{\Lambda=[2,1],R=[2,1]}_{j,\mu=[1,2]} \\ &= \frac{1}{2}\mathcal{O}^{\Lambda=[2,1],R=[2,1]}_{j=1,\mu=[1,2]} + \frac{\sqrt{3}}{2}\mathcal{O}^{\Lambda=[2,1],R=[2,1]}_{j=2,\mu=[1,2]} \\ &= \frac{1}{3\sqrt{2}}\left[\mathrm{tr}(Y)\,\mathrm{tr}(XY) - \mathrm{tr}(X)\,\mathrm{tr}(YY)\right]\end{aligned}$$

E.4 $XXYY$ for $\Lambda = [2,2]$

We have $C([2,2],[3,1],[3,1]) = C([2,2],[2,2],[2,2]) = C([2,2],[2,1,1],[2,1,1]) = 1$ and $C([2,2],[5],[5]) = C([2,2],[1^5],[1^5]) = 0$. This field content gives the highest weight state for $\Lambda = [2,2]$. Here $\Phi_r \otimes \Phi^r = X \otimes Y - Y \otimes X$.

$$\begin{aligned}\mathcal{O}^{\Lambda=[2,2],R=[2,2]} &= \frac{1}{12\sqrt{2}}\left[\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) + \mathrm{tr}(\Phi_r\Phi^r\Phi_s\Phi^s)\right] \\ \mathcal{O}^{\Lambda=[2,2],R=[3,1]} &= \frac{1}{12\sqrt{6}}\left[\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) + \mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r\Phi^s) - \mathrm{tr}(\Phi_r\Phi^r\Phi_s\Phi^s)\right] \\ \mathcal{O}^{\Lambda=[2,2],R=[2,1,1]} &= \frac{1}{12\sqrt{6}}\left[\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) - \mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r\Phi^s) - \mathrm{tr}(\Phi_r\Phi^r\Phi_s\Phi^s)\right]\end{aligned}$$

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